Ramsey Plan for Calvo's Model

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Abstract

Bellman equations for a continuation Ramsey plan and an inflation target determine a pair of infinite sequences of money creation and price level inflation rates that maximizes a benevolent time 0 government's objective function for a model of Calvo (1978). Dynamic programming provides a recursive representation of the optimal plan in which a promised inflation rate is the state variable that summarizes a continuation of a money growth sequence.

Key words: Timing protocol, Ramsey plan, Markov perfect equilibrium, Bellman equation, dynamic programming, Riccati equation.

1 Introduction

Sargent and Yang (2024) applied a brute force **machine learning** algorithm to compute an optimal government plan for a linear-quadratic version of a model of Calvo (1978). This paper instead uses ideas of Chang (1998) to formulate an optimal plan in terms of a pair of Bellman equations in the government's continuation value and a promised inflation rate. The associated dynamic program teaches us more about the structure of an optimal plan than we gathered from machine learning.[1] The promised inflation rate also equals an actual inflation rate, an inflation target, and a representative agent's expected inflation rate. We also explore how alternative timing protocols for government decision making affect outcomes.[2]

Calvo's model focuses on intertemporal tradeoffs between (a) benefits that anticipations of future deflation generate by decreasing costs of holding real money balances and thereby increasing a representative agent's **liquidity**, as measured by real money balances, and (b) costs associated with distorting taxes that a government levies to acquire the paper money that it destroys in order to generate anticipated deflation.

2 The Model

There is no uncertainty. p_t is the log of the price level at time t, m_t be the log of nominal money balances, $\theta_t = p_{t+1} - p_t$ is the net rate of inflation between t and $t+1$, and $\mu_t = m_{t+1} - m_t$ is the net rate of growth of nominal balances. The demand for real balances is governed by a discrete time version of Sargent and Wallace (1973)'s perfect foresight version of a Cagan (1956) demand function for real balances:

$$m_t - p_t = -\alpha(p_{t+1} - p_t)\,, \quad \alpha > 0 \tag{1}$$

for $t \geqslant 0$.

Equation (1) asserts that the demand for real balances is inversely related to the public's expected rate of inflation, which equals the actual rate of inflation because there is no uncertainty here.

Subtracting the demand function (1) at time t from the time $t+1$ version of this demand function gives

$$\mu_t - \theta_t = -\alpha\theta_{t+1} + \alpha\theta_t,$$

[1] In addition to ideas of Chang (1998), we will deploy the linear-quadratic dynamic programming described in chapters 5 and 19 of Ljungqvist and Sargent (2018).

[2] Sargent (2024) used the same laboratory to describe consequences of Lucas (1976).

or equivalently,

$$\theta_t = \frac{\alpha}{1+\alpha}\theta_{t+1} + \frac{1}{1+\alpha}\mu_t. \tag{2}$$

Because $\alpha > 0$, it follows that $0 < \frac{\alpha}{1+\alpha} < 1$.

Definition 2.1. For scalar b_t, let L^2 be the space of sequences $\{b_t\}_{t=0}^{\infty}$ satisfying

$$\sum_{t=0}^{\infty} b_t^2 < +\infty.$$

We say that a sequence that belongs to L^2 is *square summable*.

When we assume that the sequence $\vec{\mu} = \{\mu_t\}_{t=0}^{\infty}$ is square summable and we require that the sequence $\vec{\theta} = \{\theta_t\}_{t=0}^{\infty}$ is square summable, the linear difference equation (2) can be solved forward to get:

$$\theta_t = \frac{1}{1+\alpha}\sum_{j=0}^{\infty}\left(\frac{\alpha}{1+\alpha}\right)^j \mu_{t+j}. \tag{3}$$

Chang (1998) noted that equations (1) and (3) show that θ_t intermediates how choices of $\mu_{t+j},\ j = 0, 1, \ldots$ impinge on time t real balances $m_t - p_t = -\alpha\theta_t$. An equivalence class of continuation money growth sequences $\{\mu_{t+j}\}_{j=0}^{\infty}$ deliver the same θ_t.

We shall use this insight to simplify our analysis of alternative government policy problems. That future rates of money creation influence earlier rates of inflation makes timing protocols matter for modeling optimal government policies.

We can represent restriction (3) as

$$\begin{bmatrix} 1 \\ \theta_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1+\alpha}{\alpha} \end{bmatrix} \begin{bmatrix} 1 \\ \theta_t \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{\alpha} \end{bmatrix} \mu_t,$$

or

$$x_{t+1} = Ax_t + B\mu_t, \tag{4}$$

where

$$x_t = \begin{bmatrix} 1 \\ \theta_t \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1+\alpha}{\alpha} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -\frac{1}{\alpha} \end{bmatrix}.$$

Even though θ_0 is determined by our model and so is not an initial condition, as it ordinarily would be in the state-space model, we nevertheless write the model in the state-space form (4).

Notice that $\frac{1+\alpha}{\alpha} > 1$ is an eigenvalue of transition matrix A that threatens to destabilize the state-space system. Indeed, for arbitrary, $\vec{\mu} = \{\mu_t\}_{t=0}^{\infty}$ sequences, $\vec{\theta} = \{\theta_t\}_{t=0}^{\infty}$ will not necessarily be square summable. But the government planner will design a decision rule for μ_t that stabilizes the system and renders $\vec{\theta}$ square summable.

The government values a representative household's utility of real balances at time t according to the utility function

$$U(m_t - p_t) = u_0 + u_1(m_t - p_t) - \frac{u_2}{2}(m_t - p_t)^2, \quad u_0 > 0, u_1 > 0, u_2 > 0 \tag{5}$$

The money demand function (1) and the utility function (5) imply that

$$U(-\alpha\theta_t) = u_0 + u_1(-\alpha\theta_t) - \frac{u_2}{2}(-\alpha\theta_t)^2. \tag{6}$$

2.1 Friedman's Optimal Rate of Deflation

According to (6), the *bliss level* of real balances is $\frac{u_1}{u_2}$, and the inflation rate that attains it is:

$$\theta_t = \theta^* = -\frac{u_1}{u_2\alpha} \tag{7}$$

Milton Friedman recommended that the government withdraw and destroy money at a rate that implies an inflation rate given by (7). In our setting, that could be accomplished by setting:

$$\mu_t = \mu^* = \theta^*, \quad t \geqslant 0, \tag{8}$$

where θ^* is given by equation (7).

Milton Friedman assumed that the taxes that government imposes to collect money at rate μ_t do not distort economic decisions, e.g., they are lump-sum taxes.

2.2 Calvo's Distortion

The starting point of Calvo (1978) and Chang (1998) is that such lump-sum taxes are not available. Instead, the government acquires money by levying taxes that distort decisions and thereby impose costs on the representative consumer. The government balances the **costs** of imposing the distorting taxes against the **benefits** that expected deflation generates by raising the representative household's real money balances. Let's see how the government does that.

Via equation (3), a government plan $\vec{\mu} = \{\mu_t\}_{t=0}^{\infty}$ leads to a sequence of inflation outcomes $\vec{\theta} = \{\theta_t\}_{t=0}^{\infty}$. The government incurs social costs $\frac{c}{2}\mu_t^2$ at t when it changes the stock of nominal money balances at rate μ_t. Therefore, the one-period welfare function of a benevolent government is:

$$s(\theta_t, \mu_t) := -r(x_t, \mu_t) = \begin{bmatrix} 1 \\ \theta_t \end{bmatrix}' \begin{bmatrix} u_0 & -\frac{u_1\alpha}{2} \\ -\frac{u_1\alpha}{2} & -\frac{u_2\alpha^2}{2} \end{bmatrix} \begin{bmatrix} 1 \\ \theta_t \end{bmatrix} - \frac{c}{2}\mu_t^2 = -x_t' R x_t - Q\mu_t^2 \tag{9}$$

The government's time 0 value is

$$v_0 = -\sum_{t=0}^{\infty} \beta^t r(x_t, \mu_t) = \sum_{t=0}^{\infty} \beta^t s(\theta_t, \mu_t) \tag{10}$$

where $\beta \in (0, 1)$ is a discount factor.[3] The government's time t continuation value v_t is

$$v_t = \sum_{j=0}^{\infty} \beta^j s(\theta_{t+j}, \mu_{t+j}). \tag{11}$$

We can represent dependence of v_0 on $(\vec{\theta}, \vec{\mu})$ recursively via the difference equation

$$v_t = s(\theta_t, \mu_t) + \beta v_{t+1}. \tag{12}$$

It is useful to evaluate (12) under a time-invariant money growth rate $\mu_t = \bar{\mu}$ that, according to equation (3), would bring forth a constant inflation rate equal to $\bar{\mu}$. Under that policy,

$$v_t = V(\bar{\mu}) = \frac{s(\bar{\mu}, \bar{\mu})}{1 - \beta} \tag{13}$$

for all $t \geqslant 0$.

In summary, a representative agent's behavior as summarized by the demand function for money (1) leads to equation (3), which tells how future settings of μ affect the current value of θ. Equation (3) maps a **policy** sequence of money growth rates $\vec{\mu} = \{\mu_t\}_{t=0}^{\infty} \in L^2$ into an inflation sequence $\vec{\theta} = \{\theta_t\}_{t=0}^{\infty} \in L^2$. These in turn induce a discounted value sequence $\vec{v} = \{v_t\}_{t=0}^{\infty} \in L^2$ that satisfies recursion (12). Thus, a triple of sequences $(\vec{\mu}, \vec{\theta}, \vec{v})$ depends on a sequence $\vec{\mu} \in L^2$.

3 Three Timing Protocols

A theory of government decisions will make $\vec{\mu}$ endogenous, i.e., a theoretical **output** instead of an **input**. We consider three models of government policy. The first model

[3] We define $r(x_t, \mu_t) := -s(\theta_t, \mu_t)$ in order to represent the government's **maximization** problem in terms of our Python code for solving linear quadratic discounted dynamic programs.

describes a **Ramsey plan** chosen by a **Ramsey planner**. Here a single Ramsey planner chooses a sequence $\{\mu_t\}_{t=0}^{\infty}$ once and for all at time 0. The second model describes a **Ramsey plan** chosen by a **Ramsey planner constrained to choose a time-invariant** μ_t. Here a single Ramsey planner chooses a sequence $\{\mu_t\}_{t=0}^{\infty}$ once and for all at time 0 subject to the constraint that $\mu_t = \mu$ for all $t \geqslant 0$, The third model describes a **Markov perfect equilibrium**. Here there is a sequence of distinct policymakers indexed by $t = 0, 1, 2, \ldots$. A time t policymaker chooses μ_t only and forecasts that future government decisions are unaffected by its choice.[4]

4 A Ramsey Plan

A Ramsey planner chooses $\{\mu_t, \theta_t\}_{t=0}^{\infty}$ to maximize (10) subject to the law of motion (4). We split this problem into two stages, as in chapter 19 of Ljungqvist and Sargent (2018). In the first stage, we take the initial inflation rate θ_0 as given and solve what looks like an ordinary LQ discounted dynamic programming problem. In the second stage, we choose an optimal initial inflation rate θ_0.

Define a feasible set of $\{x_{t+1}, \mu_t\}_{t=0}^{\infty}$ sequences, with each sequence belonging to L^2:

$$\Omega(x_0) = \{x_{t+1}, \mu_t\}_{t=0}^{\infty} : x_{t+1} = Ax_t + B\mu_t, \ \forall t \geqslant 0,$$

where we require that $\{x_{t+1}, \mu_t\}_{t=0}^{\infty} \in L^2 \times L^2$.

4.1 Subproblem 1

The value function

$$J(x_0) = \max_{\{x_{t+1}, \mu_t\}_{t=0}^{\infty} \in \Omega(x_0)} \sum_{t=0}^{\infty} \beta^t s(x_t, \mu_t) \tag{14}$$

satisfies the Bellman equation:

$$J(x) = \max_{\mu, x'} \{s(x, \mu) + \beta J(x')\}$$

subject to

$$x' = Ax + B\mu$$

[4] Sargent (2024) discusses another timing protocol where there is a sequence of separate policymakers. At time t, a policymaker chooses only μ_t but believes that its choice of μ_t shapes the representative agent's beliefs about future rates of money creation and inflation, and, through them, future government actions. This is a model of a **credible government policy**, also known as a **sustainable plan**. The relationship between outcomes in this timing protocal and the first (Ramsey) timing protocol and belief structure is the subject of a literature on **sustainable** or **credible** public policies (Chari and Kehoe, 1990; Stokey, 1989, 1991)

We can map this problem into a linear-quadratic control problem and deduce an optimal value function $J(x)$.

Guessing that $J(x) = -x'Px$ and substituting into the Bellman equation gives rise to the algebraic matrix Riccati equation satisfied by P:

$$P = R + \beta A'PA - \beta^2 A'PB(Q + \beta B'PB)^{-1}B'PA$$

and an optimal decision rule:

$$\mu_t = -Fx_t$$

where:

$$F = \beta(Q + \beta B'PB)^{-1}B'PA \tag{15}$$

The value function for a (continuation) Ramsey planner is

$$v_t = -\begin{bmatrix} 1 & \theta_t \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} 1 \\ \theta_t \end{bmatrix}$$

or

$$v_t = -P_{11} - 2P_{21}\theta_t - P_{22}\theta_t^2$$

or

$$v_t = g_0 + g_1\theta_t + g_2\theta_t^2 \tag{16}$$

where

$$g_0 = -P_{11}, \quad g_1 = -2P_{21}, \quad g_2 = -P_{22}$$

The Ramsey plan for setting μ_t is

$$\mu_t = -\begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} 1 \\ \theta_t \end{bmatrix}$$

or

$$\mu_t = b_0 + b_1\theta_t \tag{17}$$

where $b_0 = -F_1, b_1 = -F_2$ and F satisfies equation (15).

The Ramsey planner's decision rule for updating θ_{t+1} is

$$\theta_{t+1} = d_0 + d_1\theta_t \tag{18}$$

where $\begin{bmatrix} d_0 & d_1 \end{bmatrix}$ is the second row of the closed-loop matrix $A - BF$ for computed in subproblem 1 above.

The linear quadratic control problem (14) satisfies regularity conditions that guarantee that $A - BF$ is a stable matrix (i.e., its maximum eigenvalue is strictly less than 1 in absolute value). Consequently, we are assured that

$$|d_1| < 1, \tag{19}$$

a stability condition that will play an important role.

It remains for us to describe how the Ramsey planner sets θ_0. Subproblem 2 does that.

4.2 Subproblem 2

The value of the Ramsey problem is

$$V^R = \max_{\theta} J(\theta)$$

We abuse notation slightly by writing $J(x)$ as $J(\theta)$ and rewrite the above equation as[5]

$$V^R = \max_{\theta_0} J(\theta_0)$$

Evidently, V^R is the maximum value of v_0 defined in equation (10).

Value function $J(\theta_0)$ satisfies

$$J(\theta_0) = -\begin{bmatrix} 1 & \theta_0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} 1 \\ \theta_0 \end{bmatrix} = -P_{11} - 2P_{21}\theta_0 - P_{22}\theta_0^2$$

The first-order necessary condition for maximizing $J(\theta_0)$ with respect to θ_0 is

$$-2P_{21} - 2P_{22}\theta_0 = 0$$

which implies

$$\theta_0 = \theta_0^R = -\frac{P_{21}}{P_{22}}$$

[5]Since $x = \begin{bmatrix} 1 \\ \theta \end{bmatrix}$, it follows that θ is the only component of x that can possibly vary.

4.3 Representation of Ramsey Plan

The preceding calculations indicate that we can represent a Ramsey plan $\vec{\mu}$ recursively with the following system created in the spirit of Chang (1998):

$$
\begin{aligned}
\theta_0 &= \theta_0^R \\
\mu_t &= b_0 + b_1 \theta_t \\
v_t &= g_0 + g_1 \theta_t + g_2 \theta_t^2 \\
\theta_{t+1} &= d_0 + d_1 \theta_t, \quad d_0 > 0, d_1 \in (0,1)
\end{aligned}
\tag{20}
$$

where b_0, b_1, g_0, g_1, g_2 are positive parameters. From condition (19), we know that $|d_1| < 1$.

To interpret system (20), think of the sequence $\{\theta_t\}_{t=0}^\infty$ as a sequence of synthetic **promised inflation rates**. For some purposes, we can think of these promised inflation rates just as computational devices for generating a sequence $\vec{\mu}$ of money growth rates that when substituted into equation (3), generate **actual** rates of inflation. It can be verified that if we substitute a plan $\vec{\mu} = \{\mu_t\}_{t=0}^\infty$ that satisfies these equations into equation (3), we obtain the same sequence $\vec{\theta}$ generated by the system (20).[6] Thus, within the Ramsey plan, **promised inflation** equals **actual inflation**.

System (20) implies that under the Ramsey plan

$$
\theta_t = d_0 \left(\frac{1 - d_1^t}{1 - d_1} \right) + d_1^t \theta_0^R, \tag{21}
$$

Because $d_1 \in (0,1)$, it follows from (21) that as $t \to \infty$, θ_t^R converges to

$$
\lim_{t \to +\infty} \theta_t^R = \theta_\infty^R = \frac{d_0}{1 - d_1}. \tag{22}
$$

Furthermore, we shall see that θ_t^R converges to θ_∞^R from above.

Meanwhile, μ_t varies over time according to:

$$
\mu_t = b_0 + b_1 d_0 \left(\frac{1 - d_1^t}{1 - d_1} \right) + b_1 d_1^t \theta_0^R. \tag{23}
$$

Variation of $\vec{\mu}^R, \vec{\theta}^R, \vec{v}^R$ over time are symptoms of time inconsistency. The Ramsey planner reaps immediate benefits from promising lower inflation later to be achieved by costly distorting taxes. These benefits are intermediated by reductions in expected inflation

[6] An application of the Big K, little k trick discussed in chapter 8 and other chapters of Ljungqvist and Sargent (2018) is at work here.

that precede the reductions in money creation rates that rationalize them, as indicated by equation (3).

5 Constrained-to-Constant-Growth-Rate Ramsey Plan

We can use brute force to create a government plan that **is** time consistent, i.e., that is time-invariant. We simply constrain a planner to choose a time-invariant money growth rate $\bar{\mu}$ so that

$$\mu_t = \bar{\mu}, \quad \forall t \geqslant 0.$$

We assume that the government knows the perfect foresight outcome implied by equation (2) that $\theta_t = \bar{\mu}$ when $\mu_t = \bar{\mu}$ for all $t \geqslant 0$. The value of the plan is

$$V(\bar{\mu}) = (1-\beta)^{-1}\left[U(-\alpha\bar{\mu}) - \frac{c}{2}(\bar{\mu})^2\right] \tag{24}$$

With the quadratic form (5) for the utility function U, the maximizing $\bar{\mu}$ is

$$\mu^{CR} = \max_{\bar{\mu}} V(\bar{\mu}) = -\frac{\alpha u_1}{\alpha^2 u_2 + c}. \tag{25}$$

The optimal value attained by a **constrained to constant** μ Ramsey planner is

$$V(\mu^{CR}) \equiv V^{CR} = (1-\beta)^{-1}\left[U(-\alpha\mu^{CR}) - \frac{c}{2}(\mu^{CR})^2\right]. \tag{26}$$

6 Markov Perfect Governments

One can assume another timing protocol in order to render government decisions time-consistent. Consider a sequence of government policymakers. A time t government chooses μ_t and expects all future governments to set $\mu_{t+j} = \bar{\mu}$. When it sets μ_t, a government at t believes that $\bar{\mu}$ is unaffected by its choice of μ_t.

According to equation (3), the time t rate of inflation is then:

$$\theta_t = \frac{1}{1+\alpha}\mu_t + \frac{\alpha}{1+\alpha}\bar{\mu}, \tag{27}$$

which expresses inflation θ_t as a geometric weighted average of the money growth today μ_t and money growth from tomorrow onward $\bar{\mu}$.

Given $\bar{\mu}$, the time t government chooses μ_t to maximize:

$$H(\mu_t, \bar{\mu}) = U(-\alpha\theta_t) - \frac{c}{2}\mu_t^2 + \beta V(\bar{\mu}) \tag{28}$$

where $V(\bar{\mu})$ is given by formula (13) for the time 0 value v_0 of recursion (12) under a money supply growth rate that is forever constant at $\bar{\mu}$.

Substituting (27) into (28) and expanding gives:

$$\begin{aligned} H(\mu_t, \bar{\mu}) = u_0 + u_1 \left(-\frac{\alpha^2}{1+\alpha}\bar{\mu} - \frac{\alpha}{1+\alpha}\mu_t \right) - \frac{u_2}{2} \left(-\frac{\alpha^2}{1+\alpha}\bar{\mu} - \frac{\alpha}{1+\alpha}\mu_t \right)^2 \\ - \frac{c}{2}\mu_t^2 + \beta V(\bar{\mu}) \end{aligned} \tag{29}$$

The first-order necessary condition for maximizing $H(\mu_t, \bar{\mu})$ with respect to μ_t is:

$$-\frac{\alpha}{1+\alpha}u_1 - u_2(-\frac{\alpha^2}{1+\alpha}\bar{\mu} - \frac{\alpha}{1+\alpha}\mu_t)(-\frac{\alpha}{1+\alpha}) - c\mu_t = 0$$

Rearranging we get the time t government's best response map

$$\mu_t = f(\bar{\mu})$$

where

$$f(\bar{\mu}) = \frac{-u_1}{\frac{1+\alpha}{\alpha}c + \frac{\alpha}{1+\alpha}u_2} - \frac{\alpha^2 u_2}{\left[\frac{1+\alpha}{\alpha}c + \frac{\alpha}{1+\alpha}u_2\right](1+\alpha)}\bar{\mu}$$

A *Markov Perfect Equilibrium* (MPE) outcome μ^{MPE} is a fixed point of the best response map:

$$\mu^{MPE} = f(\mu^{MPE})$$

Calculating μ^{MPE}, we find

$$\mu^{MPE} = -\frac{\alpha u_1}{\alpha^2 u_2 + (1+\alpha)c}. \tag{30}$$

The value of a MPE is:

$$V^{MPE} = \frac{s(\mu^{MPE}, \mu^{MPE})}{1-\beta} \tag{31}$$

or $V^{MPE} = V(\mu^{MPE})$, where $V(\cdot)$ is given by formula (13).

Under the Markov perfect timing protocol, a government takes $\bar{\mu}$ as given when it chooses μ_t, and we equate $\mu_t = \mu$ only **after** we have computed a time t government's first-order condition for μ_t.

7 Outcomes under Three Timing Protocols

Let's compare outcome sequences $\{\theta_t, \mu_t\}$ under three timing protocols associated with (1) a standard Ramsey plan with its time-varying $\{\theta_t, \mu_t\}$ sequences, (2) a Markov perfect equilibrium, with its time-invariant $\{\theta_t, \mu_t\}$ sequences, and (3) a Ramsey plan in which the planner is restricted to choose a time-invariant $\mu_t = \mu$ for all $t \geqslant 0$.

Figure 1 plots policy functions for a continuation Ramsey planner with $\beta = 0.75, c = 2$ and $\alpha = 1, u_0 = 1, u_1 = 0.5, u_2 = 3$. The dotted line is the 45-degree line. The blue line shows

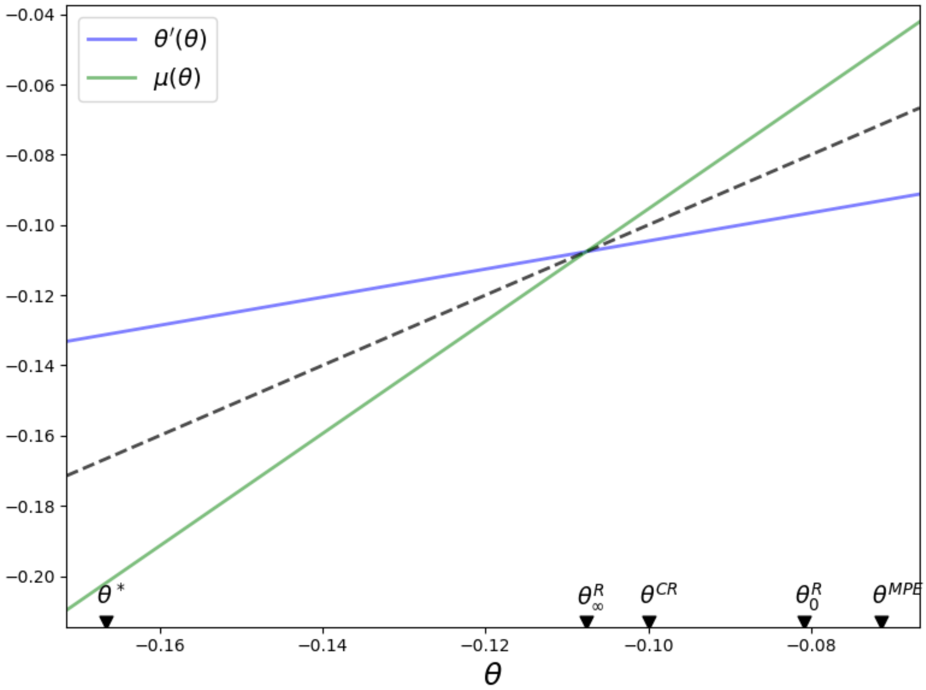

Figure 1: μ_t and θ_{t+1} as functions of θ

the choice of $\theta_{t+1} = \theta'$ chosen by a continuation Ramsey planner who inherits $\theta_t = \theta$. The green line shows a continuation Ramsey planner's choice of $\mu_t = \mu$ as a function of an inherited $\theta_t = \theta$.

Dynamics under the Ramsey plan are confined to $\theta \in \left[\theta^R_\infty, \theta^R_0\right]$. The blue and green lines intersect each other and the 45-degree line at $\theta = \theta^R_\infty$.

Notice that for $\theta \in \left(\theta^R_\infty, \theta^R_0\right]$

- $\theta' < \theta$ because the blue line is below the 45-degree line, and
- $\mu > \theta$ because the green line is above the 45-degree line.

It follows that under the Ramsey plan $\{\theta_t\}$ and $\{\mu_t\}$ both converge monotonically from above to θ^R_∞.

The orange curve in Figure 2 plots the Ramsey planner's value function $J(\theta)$. We know that $J(\theta)$ is maximized at θ^R_0, the best time 0 promised inflation rate. The figure indicates the limiting value θ^R_∞, the limiting value of promised inflation rate θ_t under the Ramsey plan as $t \to +\infty$. It also indicates an MPE inflation rate θ^{MPE}, the inflation θ^{CR} under a Ramsey plan constrained to a constant money creation rate, and a bliss inflation θ^*. Notice that:

$$\theta^* < \theta_\infty^R < \theta^{CR} < \theta_0^R < \theta^{MPE},$$

which suggests that

- $\theta_0^R < \theta^{MPE}$: the MPE inflation rate exceeds the initial Ramsey inflation rate,
- $\theta_\infty^R < \theta^{CR} < \theta_0^R$: the initial Ramsey deflation rate, and the associated tax distortion cost $c\mu_0^2$ is less than the limiting Ramsey inflation rate θ_∞^R and the associated tax distortion cost μ_∞^2,
- $\theta^* < \theta_\infty^R$: the limiting Ramsey inflation rate exceeds the bliss level of inflation

The blue curve in Figure 2 plots the value function of a Ramsey planner who is constrained to choose a constant μ. Since a time-invariant μ implies a time-invariant θ, we take the liberty of labeling this value function $V(\theta)$. The graph reveals interesting relationships between the value functions $J(\theta) \geqslant V(\theta)$.

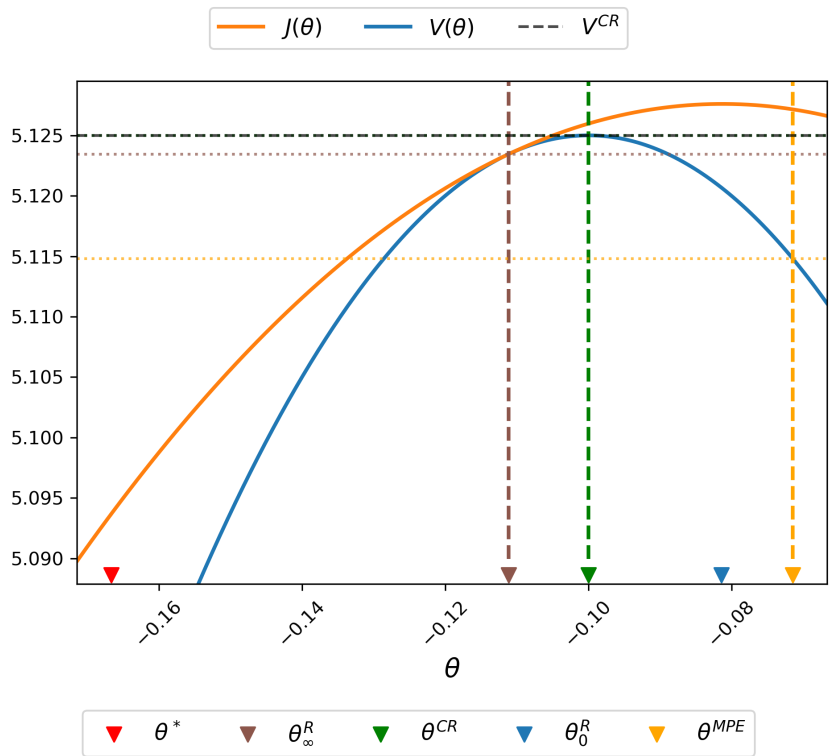

Figure 2: Value functions J and V over θ

Furthermore, notice that

- the orange $J(\theta)$ value function lies above the blue V value function except at $\theta = \theta_\infty^R$;

- the maximizer θ_0^R of $J(\theta)$ occurs at the top of the orange curve;
- the maximizer θ^{CR} of $V(\theta)$ occurs at the top of the blue curve;
- $J(\theta_\infty^R) = V(\theta_\infty^R)$;
- the "timeless perspective" inflation and money creation rate θ_∞^R occurs where $J(\theta)$ is tangent to $V(\theta)$;
- the Markov perfect inflation and money creation rate θ^{MPE} exceeds θ_0^R;
- the value $V(\theta^{MPE})$ of the Markov perfect rate of money creation rate θ^{MPE} is less than the value $V(\theta_\infty^R)$ of the worst continuation Ramsey plan; and
- the continuation value $J(\theta^{MPE})$ of the Markov perfect rate of money creation rate θ^{MPE} is greater than the value $V(\theta_\infty^R)$ and of the continuation value $J(\theta_\infty^R)$ of the worst continuation Ramsey plan.

8 Plausibility of a Ramsey Plan?

Many economists doubt the relevance of a timing protocol in which a plan for setting a sequence of policy variables is chosen once and for all at time 0. They instead prefer the sequential timing protocol that prevails in a Markov perfect equilibrium. But there are superior plans that, like a Markov perfect equilibrium, provide no incentives to deviate from the plan. Research of Abreu (1988), Chari and Kehoe (1990), Stokey (1989), and Stokey (1991) applied by Sargent (2024) in the present context described conditions under which a Ramsey plan emerges from a sequential timing protocol for government decision makers. To accomplish that, it is necessary to expand a description of a plan to include a coherent set of beliefs that deter deviating from it.

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