

Implementing a Ramsey Plan*

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Abstract

Lucas and Stokey (1983) motivated future governments to confirm an optimal tax plan by rescheduling government debt appropriately. Debortoli et al. (2021) showed that sometimes that does not work. We show how a Ramsey plan can always be implemented by adding instantaneous debt to Lucas and Stokey's contractible subspace and requiring that each continuation government preserve that debt's purchasing power instantaneously. We formulate the Ramsey problem with a Bellman equation and use it to study settings with various initial term debt structures and government spending processes. We extract implications about tax smoothing and effects of fiscal policies on bond markets.

Keywords: Ramsey planner, continuation Ramsey planner, implementability, instantaneous debt, local commitment, dynamic programming.

JEL Classification:

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1 Introduction

Before Lucas and Stokey (1983), Lucas (1972a,b, 1975) had modeled government policies as exogenous stochastic processes.¹ Kydland and Prescott (1977) and Calvo (1978) had convinced Lucas that Ramsey plans are implausible: under sequential timing protocols that actually characterize policy-making processes, they are not implementable. Although Kydland and Prescott (1980) offered a concise way of representing Ramsey plans recursively, those plans were still not implementable. Then Lucas and Stokey (1983) showed how to implement Ramsey plans under a plausible sequential timing protocol. That opened the door to using rational expectations model to analyze optimal government policy in settings with plausible timing protocols.

As their laboratory, Lucas and Stokey turned to Barro (1979), whose principal conclusion was that intertemporal properties of an optimal tax rate are completely disconnected from the intertemporal properties of the exogenous government expenditure process that is to be financed. Lucas and Stokey set out to understand sources of that striking tax-smoothing outcome. To do so, they formulated a Ramsey plan within a complete-markets environment with a distorting flat rate tax and described a system of financial obligations sufficient to induce a sequence of governments to confirm a Ramsey plan for tax rates. By constructing a Ramsey plan that is implementable (i.e., time-consistent) under particular assumptions about what inherited promises governments are bound to respect, Lucas and Stokey responded to Kydland and Prescott (1977)'s and Calvo (1978)'s doubts about the plausibility of a Ramsey plan. In addition, their closed-economy, complete-markets analysis replaced Barro (1979)'s tax-rate, government expenditure disconnection finding with outcomes in which intertemporal properties of the tax rate mirror those of government expenditures.²

This paper resolves a difficulty that Debortoli, Nunes, and Yared (2021) detected with the prescription for an optimal term structure of government debt proposed by Lucas and Stokey (1983, Sec. 3). In Lucas and Stokey's model, each of a sequence of governments is required to finance an exogenous and immutable joint stochastic process for government expenditures $\{G_t\}_{t=0}^{\infty}$ and debt service coupons $\{b_{0,t}\}_{t=0}^{\infty}$. At time 0, a Ramsey planner

¹Lucas (1978) had taken a representative agent's consumption process and the associated stochastic discount factor as exogenous. A key aspect of the Lucas and Stokey (1983) analysis was to design a tax plan to manipulate a stochastic discount factor process.

²Subsection 7.3 describes a special case of Lucas and Stokey's model in which Barro's disconnection result prevails.

chooses a process of distorting flat rate taxes and possibly a restructured debt service coupon process $\{\hat{b}_{0,t}\}_{t=0}^{\infty}$. At times $t > 0$, continuation Ramsey planners are free to redesign the flat rate tax process and to reschedule government debt from t onward, but they must honor the continuation debt service coupon process that they inherited. Lucas and Stokey provided examples in which appropriate term structures of government debt induce continuation planners to choose to continue the original Ramsey tax rate process. However, Debortoli, Nunes, and Yared (2021) constructed examples in which Lucas and Stokey’s way of restructuring government debt fails to induce continuation planners to confirm the Ramsey plan.

To set the stage for our expansion of Lucas and Stokey’s contractible subspace, it is useful to read how Aguiar et al. (2019) contrasted Lucas and Stokey’s (1983) model with theirs:

Lucas and Stokey (1983) studied optimal fiscal policy with complete markets and discussed at length how maturity choice is a useful tool to provide incentives to a government that lacks commitment to taxes and debt issuance, but cannot default. The government has an incentive to manipulate the risk-free real interest rate, by changing taxes which affects investors’ marginal utility, to alter the value of outstanding long-term bonds, something ruled out by our small open-economy framework with risk-neutral investors. Their main result is that the maturity of debt should be spread out, resembling the issuance of consols. Our model instead emphasizes default risk, something absent from their work. Our main result is also the reverse, providing a force for the exclusive use of short-term debt.

Short-term debt also plays an essential role in our model. But unlike Aguiar et al., we retain almost all other parts of Lucas and Stokey’s and Debortoli et al.’s structure, including complete markets, a closed economy setting, the presence of incentives that continuation Ramsey planners have to use flat rate taxes to manipulate interest rates, and obligations of governments to honor all government debt that they inherit. To Lucas and Stokey’s and Debortoli et al.’s contractible subspace for debt restructurings $\{\hat{b}_{0,t}\}_{t=0}^{\infty}$, we add an instantaneous debt balance \hat{B}_t that time- t continuation Ramsey planners must service, together with a local commitment condition. By managing \hat{B}_t appropriately, Lucas and Stokey’s Ramsey plan can be implemented without restructuring $\{b_{0,t}\}_{t=0}^{\infty}$, i.e., by setting $\{\hat{b}_{0,t}\}_{t=0}^{\infty} = \{b_{0,t}\}_{t=0}^{\infty}$. (A Ramsey plan that can be implemented is typically described as

being “time consistent.”)³ See Table 1 to see how the contractible subspace in our model compares with Lucas and Stokey’s and Debortoli et al.’s.

Table 1: Comparison with existing methods

	Contractible space	Implementable?
Lucas and Stokey (1983)	$\{\hat{b}_{0,t}\}_{t=0}^\infty \neq \{b_{0,t}\}_{t=0}^\infty$	Not always
Debortoli, Nunes, and Yared (2021)	$\{\hat{b}_{0,t}\}_{t=0}^\infty \neq \{b_{0,t}\}_{t=0}^\infty$	Not always
This paper	$\hat{B}_t, \{\hat{b}_{0,t}\}_{t=0}^\infty = \{b_{0,t}\}_{t=0}^\infty$ Local commitment	Always

We begin by noting that a bond price process $\{q_{0,u}^*; u \geq 0\}$ and a primary surplus process $\{S_{0,u}^*; u \geq 0\}$ are affiliated with Lucas and Stokey’s Ramsey plan. Together with the initial term debt structure $\{b_{0,u}; u \geq 0\}$, these two objects uniquely determine a process: $\Pi_t^* = \int_0^t \frac{q_{0,u}^*}{q_{0,t}^*} (b_{0,u} - S_{0,u}^*) du$ that accumulates the government’s unpaid liabilities from time 0 to t , conditional on the government not having rescheduled the initial term debt process $\{b_{0,t}\}_{t=0}^\infty$ before time t . Time u flow liability $(b_{0,u} - S_{0,u}^*)$ times price $\frac{q_{0,u}^*}{q_{0,t}^*}$ contributes to a time t value of government liabilities. Our implementation of a Ramsey plan abstains from restructuring the initial debt term structure $\{b_{0,t}\}_{t=0}^\infty$ and instead instructs continuation governments to issue only instantaneous debt that has infinitesimal maturity. This arrangement takes into account all relevant quantity-and-equilibrium-price information about the initial term debt structure that concerns the Ramsey planner.

The Ramsey planner thus implements its plan by leaving the initial debt term structure untouched and paying for all government purchases and debt servicing by issuing instantaneous debt. Continuation governments are obligated to preserve the “purchasing power” of cumulative government’ liabilities Π_t^* , defined as the product of Π_t^* and the representative household’s marginal utility of consumption $U_{C,t}^*$. The Ramsey planner leaves each future government an instantaneous debt balance \hat{B}_t that equals Π_t^* , the purchasing power of which continuation governments must preserve. But continuation governments

³Ljungqvist and Sargent (2018, ch. 20) assume that the government has access to only one-period debt in a discrete time version of Lucas and Stokey’s model. It is a counterpart to instantaneous debt in our continuous time model. It also leads to Bellman equations that can be used to express a discrete-time counterpart to the local commitment constraint that we impose in this paper.

are tempted to deviate from the Ramsey tax plan in order to manipulate that purchasing power. Therefore, to implement a Ramsey plan we augment Lucas and Stokey’s assumptions by adding a local commitment condition that requires that continuation governments preserve the purchasing power of instantaneous government debt. Sections 4 and 5 show that this restriction is sufficient to induce them to confirm the Ramsey plan.

Sometimes our implementation and Debortoli et al.’s implementation both work, for example when initial debt servicing costs are not too high. But they have different implications for the behavior of the government’s “gross of interest surplus,” i.e., the government’s primary surplus plus all debt service payments. In the context of Debortoli et al.’s example, subsection 5.2 shows that in Lucas and Stokey’s implementation the government’s gross of interest surplus always equals zero, while in ours the government runs a (positive) gross of interest deficit when term debt $b_{0,t}$ is high initially, then runs a zero gross of interest deficit after term debt payments $b_{0,t}$ drops to zero. This is because the government’s term debt is restructured in Lucas and Stokey’s implementation, but not in ours. Our implementation connects well to Barro’s tax smoothing recommendation and is reminiscent of Milton Friedman’s permanent-income hypothesis.

Sections 6 and 7 describe a recursive formulation of the Ramsey problem for general right continuous with left limit (Càdlàg) $\{b_{0,t}, G_t\}$ processes. A key state variable X_t equals the product of the government’s cumulative liability Π_t and contemporaneous marginal utility $U_{C,t}$. The value function is additively separable in X_t and time t and so can be represented as $-\Phi_0^* X_t + H(t; \Phi_0^*)$, where Φ_0^* is the multiplier on the implementability constraint in a Lagrangian formulation.

Sections 8 and 9 use our recursive formulations of the Ramsey problem to implement Ramsey plan for general Càdlàg $\{b_{0,t}, G_t\}$ processes. In particular, when $\{b_{0,t}, G_t\}$ are continuous functions of time, the local commitment condition that facilitates implementing the Ramsey plan simply requires that the consumption rate is continuous in time. Furthermore, Ramsey plans for settings with general Càdlàg $\{b_{0,t}, G_t\}$ processes are well approximated by limits of Ramsey plans with appropriate continuous $\{b_{0,t}, G_t\}$ processes.

2 Environment

Time $t \in [0, \infty)$ is continuous. A representative household and benevolent government participate in a complete set of perfectly competitive markets. The government finances an exogenous stream of expenditures with a stream of distorting flat tax rates. There is

no uncertainty. At time 0, a Ramsey planner selects the best competitive equilibrium with distorting taxes.

Assumption 1. *Exogenous flows of government expenditures $\vec{G}_0 = \{G_t; t \geq 0\}$ and debt-service payouts $\vec{b}_0 = \{b_{0,t}; t \geq 0\}$ are Càdlàg processes, i.e., $\lim_{s \downarrow t} b_{0,s} = b_{0,t}$ and $\lim_{s \downarrow t} G_s = G_t$ for all $t \geq 0$; and $\lim_{s \uparrow t} b_{0,s}$ and $\lim_{s \uparrow t} G_s$ exist for all $t > 0$.*

The representative household supplies a labor process $\vec{N}_0 = \{N_{0,t}; t \geq 0\}$ that produces a flow of a single nonstorable good that can be divided between a consumption flow $\vec{C}_0 = \{C_{0,t}; t \geq 0\}$ and a government expenditure flow \vec{G}_0 :

$$C_{0,t} + G_t = N_{0,t}, \text{ for all } t \geq 0. \quad (1)$$

At time 0, the representative household orders consumption and labor supply streams $\{C_{0,t}; t \geq 0\}$ and $\{N_{0,t}; t \geq 0\}$ according to

$$\int_0^\infty e^{-\rho t} U(C_{0,t}, N_{0,t}) dt, \quad (2)$$

where $\rho > 0$ and $U(\cdot, \cdot)$ is strictly increasing in consumption C , strictly decreasing in labor supply N , globally concave, and continuously differentiable. Since consumption and labor decisions are convex controls and the household derives utility $U(C_{0,t}, N_{0,t}) dt$ over a small time interval dt , we focus on policies that we call admissible in which \vec{C}_0 and \vec{N}_0 are Càdlàg processes. Along with Debortoli, Nunes, and Yared (2021), we assume that labor supply $N_{0,t}$ has no upper bound.

Let $\tau_{0,t}$ denote a tax rate at t chosen at time 0 and $q_{0,t}$ be the time-0 value of a zero-coupon bond with a unit payoff at t . A first subscript denotes the time that a variable is chosen and that a second subscript denotes a time that the variable is realized.

The representative household faces a single intertemporal budget constraint:

$$\int_0^\infty C_{0,t} q_{0,t} dt \leq \int_0^\infty b_{0,t} q_{0,t} dt + \int_0^\infty (1 - \tau_{0,t}) N_{0,t} q_{0,t} dt. \quad (3)$$

The left side of (3) is the present value of the household's consumption stream $\{C_{0,t}; t \geq 0\}$ and the right side is the sum of the household's financial wealth and its human wealth in the form of the present value of after-tax labor income. Given tax rate process $\{\tau_{0,t}; t \geq 0\}$ and bond price process $\{q_{0,t}; t \geq 0\}$, the household chooses $\{C_{0,t}, N_{0,t}; t \geq 0\}$ to maximize (2) subject to constraint (3). That optimum problem brings the following first-order necessary

conditions:

$$1 - \tau_{0,t} = -\frac{U_{N,t}}{U_{C,t}}, \quad (4)$$

$$q_{0,t} = e^{-\rho t} \frac{U_{C,t}}{U_{C,0}}. \quad (5)$$

The time-0 government finances an initial debt and its exogenous spending stream by taxing labor income:

$$\int_0^\infty (\tau_{0,t} N_{0,t} - G_t) q_{0,t} dt \geq \int_0^\infty b_{0,t} q_{0,t} dt. \quad (6)$$

Let $S_{0,t} = \tau_{0,t} N_{0,t} - G_t$ denote the primary surplus. The left side is time 0 value of $\{S_{0,t}; t \geq 0\}$ and the right side is time 0 value of payouts on government debt: $\{b_{0,t}; t \geq 0\}$.

Definition 1. *Given the government's initial debt term structure $\{b_{0,t}; t \geq 0\}$ and spending flow process $\{G_t; t \geq 0\}$, a competitive equilibrium is a feasible allocation $\{C_{0,t}, N_{0,t}; t \geq 0\}$, a flat rate tax process $\{\tau_{0,t}; t \geq 0\}$, and a bond price process $\{q_{0,t}; t \geq 0\}$ for which*

- *the government's budget constraint (6) is satisfied, and*
- *given the government spending, tax, and bond price process, the allocation solves the household's optimization problem*

We follow Lucas and Stokey (1983) and use first-order conditions (4) and (5) together with feasibility constraint (1) to eliminate tax rates and bond prices from the government budget constraint. We thereby obtain the following implementability constraint on competitive equilibrium allocations $\{C_{0,t}, N_{0,t}\}_{t=0}^\infty$:

$$\int_0^\infty e^{-\rho t} (C_{0,t} U_{C,t} + (C_{0,t} + G_t) U_{N,t}) dt \geq \int_0^\infty e^{-\rho t} b_{0,t} U_{C,t} dt. \quad (7)$$

Proposition 1. *Given an initial term-debt structure $\{b_{0,t}; t \geq 0\}$ and government spending path $\{G_t; t \geq 0\}$, an allocation $\{C_{0,t}, N_{0,t}; t \geq 0\}$ is a competitive equilibrium allocation if and only if it satisfies (1) for all $t \geq 0$ and (7) at $t = 0$.*

Ramsey problem. A Ramsey planner chooses a Càdlàg process $\vec{C}_0 = \{C_{0,t}; t \geq 0\}$ that maximizes

$$\int_0^\infty e^{-\rho t} U(C_{0,t}, C_{0,t} + G_t) dt \quad (8)$$

subject to implementability constraint (7). By using first-order conditions (4) and (5), we can compute an associated tax rate process and bond price system. After the Ramsey planner chooses a tax plan and associated price system, the representative agent's best responses confirm the associated competitive equilibrium allocation.

We next turn to how Lucas and Stokey's Ramsey planner reschedules an initial debt structure $\{b_{0,t}\}_{t=0}^{\infty}$ to another debt structure $\{\hat{b}_{0,t}\}_{t=0}^{\infty} \neq \{b_{0,t}\}_{t=0}^{\infty}$, hoping to induce future planners to confirm the Ramsey plan it has designed.

Lucas and Stokey's continuation debt structure

To begin, we note that at a Ramsey allocation $\{C_{0,t}^*, N_{0,t}^*; t \geq 0\}$, flat tax rate process $\{\tau_{0,t}^*; t \geq 0\}$, and price system $\{q_{0,t}^*; t \geq 0\}$, many possible debt structures $\{\hat{b}_{0,t}\}_{t=0}^{\infty}$ satisfy time 0 and time $t > 0$ continuation government budget constraints:

$$\int_0^{\infty} \hat{b}_{0,s} q_{0,s}^* ds \leq \int_0^{\infty} (\tau_{0,s}^* N_{0,s}^* - G_s) q_{0,s}^* ds \quad (9)$$

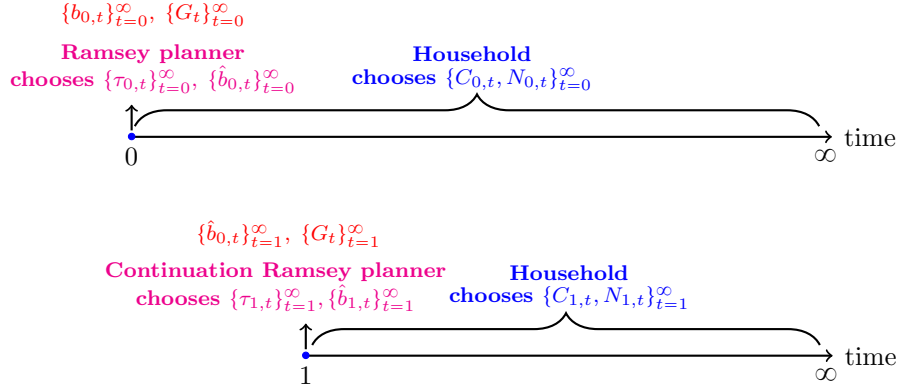
$$\int_t^{\infty} \hat{b}_{0,s} q_{0,s}^* ds \leq \int_t^{\infty} (\tau_{0,s}^* N_{0,s}^* - G_s) q_{0,s}^* ds. \quad (10)$$

Definition 2. *A continuation of a Ramsey plan at $t > 0$ is the tail $\{C_{0,s}^*, N_{0,s}^*, \tau_{0,s}^*, q_{0,s}^*; s \geq t\}$ of a Ramsey plan for $s \geq t$. A continuation Ramsey planner at time $t > 0$ must honor the (inherited) continuation debt structure $\{\hat{b}_{0,s}\}_{s=t}^{\infty}$ and chooses a continuation plan $\{\tau_{t,s}\}_{s=t}^{\infty}$ for tax rates that need not equal the continuation of the original Ramsey plan, $\{\tau_{0,s}^*\}_{s=t}^{\infty}$, and a new debt structure $\{\hat{b}_{t,s}\}_{s=t}^{\infty}$. A Ramsey plan is said to be implemented (or "time consistent") when all continuation Ramsey planners (i.e., for all $t > 0$) choose to confirm it.*

Figure 1 illustrates the timing protocol used by Debortoli, Nunes, and Yared (2021). At time 0, a Ramsey planner confronts a government purchase stream $\{G_t\}_{t=0}^{\infty}$ and an initial debt structure $\{b_{0,t}\}_{t=0}^{\infty}$ that it must finance with a sequence of flat rate taxes $\{\tau_{0,t}\}_{t=0}^{\infty}$ and possibly restructured debt $\{\hat{b}_{0,t}\}_{t=0}^{\infty}$. Knowing the tax and debt structure sequences, at time 0 the representative agent chooses $\{C_{0,t}, N_{0,t}\}_{t=0}^{\infty}$. In Debortoli et al.'s laboratory, a single continuation Ramsey planner at time $t = 1$ confronts $\{G_t\}_{t=1}^{\infty}$ and $\{\hat{b}_{0,t}\}_{t=1}^{\infty}$ that it must finance with a time $t = 1$ continuation tax plan $\{\tau_{1,t}\}_{t=1}^{\infty}$.

Lucas and Stokey (1983) constructed examples in which budget-feasible debt structures $\{\hat{b}_{0,t}\}_{t=1}^{\infty}$ induce a continuation Ramsey planner to confirm a continuation of a Ramsey tax plan $\{\tau_{0,t}^*; t \geq 1\}$ and its associated allocation and price system: $\{C_{0,t}^*, N_{0,t}^*, q_{0,t}^*; t \geq 1\}$.

Figure 1: Timing protocol for Ramsey planner and time 1 continuation Ramsey planner.



However, Debortoli, Nunes, and Yared (2021) constructed examples for which debt term structure $\{\hat{b}_{0,t}\}_{t=1}^{\infty}$ along lines recommended by Lucas and Stokey (1983) fails to induce a continuation Ramsey planner to confirm a continuation of the Ramsey plan. To set the stage for a continuous time version of Debortoli et al.'s counterexample, the next section presents the Ramsey planner's Lagrangian.

3 Ramsey Planner's Lagrangian

Attach a multiplier Λ_0 to implementability constraint (7) and form a Lagrangian:

$$\mathcal{L}_0 = \int_0^{\infty} e^{-\rho t} [U(C_{0,t}, C_{0,t} + G_t) + \Lambda_0 (C_{0,t}U_{C,t} + (C_{0,t} + G_t)U_{N,t} - b_{0,t}U_{C,t})] dt. \quad (11)$$

The Ramsey planner maximizes the right side of (11) over consumption plans $\{C_{0,t}\}_{t=0}^{\infty}$ and minimizes over the nonnegative multiplier Λ_0 . Call the extremizing values: $\{C_{0,t}^*\}_{t=0}^{\infty}$ and Λ_0^* . For a given Λ_0 , the optimal consumption plan satisfies⁴

$$C(\Lambda_0; b_{0,t}, G_t) := \arg \max_{C_{0,t}} [U(C_{0,t}, C_{0,t} + G_t) + \Lambda_0 (C_{0,t}U_C + (C_{0,t} + G_t)U_N - b_{0,t}U_C)]. \quad (12)$$

Substituting (12) into implementability condition (7), we deduce that Λ_0 must satisfy:

$$\int_0^{\infty} e^{-\rho t} [C(\Lambda_0; b_{0,t}, G_t)U_C + (C(\Lambda_0; b_{0,t}, G_t) + G_t)U_N] dt = \int_0^{\infty} e^{-\rho t} b_{0,t}U_C dt. \quad (13)$$

⁴We assume a regularity condition under which the optimal consumption plan for the problem defined in (12) is unique for a given Λ_0 .

Substituting $C(\Lambda_0; b_{0,t}, G_t)$ into the right side of (11) lets us write \mathcal{L}_0 as a function $\mathcal{L}_0(\Lambda_0)$. An optimal Λ_0^* is the non-negative root of equation (13) that maximizes $\mathcal{L}_0(\Lambda_0)$. In sum, the Ramsey allocation is given by: $C_{0,t}^* = C(\Lambda_0^*; b_{0,t}, G_t)$ and $N_{0,t}^* = C_{0,t}^* + G_t = N(\Lambda_0^*; b_{0,t}, G_t)$.

Proposition 2. *Under regularity conditions provided in Appendix A, there exists a non-negative root Λ_0^* of equation (13) that solves*

$$\Lambda_0^* = \arg \min_{\Lambda_0} \mathcal{L}_0(\Lambda_0),$$

where the Ramsey allocation, tax rate, and bond price process satisfy

$$\begin{aligned} C_{0,t}^* &= C(\Lambda_0^*; b_{0,t}, G_t); & N_{0,t}^* &= C_{0,t}^* + G_t; \\ \tau_{0,t}^* &= 1 + \frac{U_{N,t}(C_{0,t}^*, N_{0,t}^*)}{U_{C,t}(C_{0,t}^*, N_{0,t}^*)}; & q_{0,t}^* &= e^{-\rho t} \frac{U_{C,t}(C_{0,t}^*, N_{0,t}^*)}{U_{C,0}(C_{0,0}^*, N_{0,0}^*)}. \end{aligned}$$

The value \mathcal{L}_0^* of a Ramsey plan is $\int_0^\infty e^{-\rho t} U(C_{0,t}^*, C_{0,t}^* + G_t) dt$.

The primary surplus under the Ramsey plan is

$$S_{0,t}^* = S(C_{0,t}^*) = \tau_{0,t}^* N_{0,t}^* - G_t. \quad (14)$$

The time- t value of cumulative deficits in the absent of any rescheduling the initial term debt process $\{b_{0,t}\}$ is

$$\Pi_t^* = \frac{1}{q_{0,t}^*} \int_0^t q_{0,u}^* (b_{0,u} - S(C_{0,u}^*)) du, \quad (15)$$

where the increment to the government's IOU's over a small du interval is $(b_{0,u} - S(C_{0,u}^*)) du$ and $q_{0,u}^*/q_{0,t}^*$ converts a time u deficit to its time t value. The stock Π_t^* plays a crucial role in our way of implementing the Ramsey plan.

4 The Counterexample

To construct a continuous-time version of Debortoli et al.'s counterexample to Lucas and Stokey's implementation of a Ramsey plan, we adopt the special instantaneous utility function:

$$U(C, N) = \log C - \eta \frac{N^\gamma}{\gamma}, \quad (16)$$

where $\eta > 0$ and $\gamma \geq 1$. The Ramsey planner must finance the time-invariant government expenditure process $G_t = G$ for all $t \geq 0$ and an initial debt structure:

$$b_{0,t} = \begin{cases} b_0 > 0 & t \in [0, T), \\ 0 & t \geq T, \end{cases} \quad (17)$$

for a given $T > 0$. To ease exposition, for the remainder of the paper, we'll work with the $\gamma = 1$ case where $U_C = 1/C$ and $U_N = -1$. The tax rate then satisfies $\tau_{0,t} = 1 - \eta C_{0,t}$ for all $t \geq 0$ and the primary surplus $S_{0,t}$ given in (14) takes the form of:

$$S_{0,t} = S(C_{0,t}) = C_{0,t} (1 - \eta(C_{0,t} + G_t)) . \quad (18)$$

Next, we state a continuous-time version of Lemma 3 in Debortoli, Nunes, and Yared (2021) for the $\gamma = 1$ case.

Lemma 1. *When b_0 is not too high, a Ramsey plan exists and is*

$$C_{0,t}^* = C_0^* \mathbf{1}_{\{t \in [0, T)\}} + C_1^* \mathbf{1}_{\{t \in [T, \infty)\}}; \quad N_{0,t}^* = N_0^* \mathbf{1}_{\{t \in [0, T)\}} + N_1^* \mathbf{1}_{\{t \in [T, \infty)\}}, \quad (19)$$

$$\tau_{0,t}^* = \tau_0^* \mathbf{1}_{\{t \in [0, T)\}} + \tau_1^* \mathbf{1}_{\{t \in [T, \infty)\}}; \quad q_{0,t}^* = e^{-\rho t} (\mathbf{1}_{\{t \in [0, T)\}} + C_0^*/C_1^* \mathbf{1}_{\{t \in [T, \infty)\}}), \quad (20)$$

where $\mathbf{1}_{\{A\}}$ is an indicator function that equals one if the event A occurs and zero otherwise,

$$C_0^* = C(\Lambda_0^*; b_0, G) = \frac{2b_0\Lambda_0^*}{\sqrt{1 + 4\eta b_0\Lambda_0^*(1 + \Lambda_0^*)} - 1}, \quad C_1^* = C(\Lambda_0^*; 0, G) = \frac{1}{\eta(1 + \Lambda_0^*)}, \quad (21)$$

$N_0^* = C_0^* + G$, and $N_1^* = C_1^* + G$. The Ramsey tax plan is given by $\tau_0^* = 1 - \eta C_0^*$ and $\tau_1^* = 1 - \eta C_1^*$. Finally, Λ_0^* is the unique non-negative root of

$$\frac{1 - e^{-\rho T}}{\rho} \left(1 - \eta(C(\Lambda_0^*; b_0, G) + G) - \frac{b_0}{C(\Lambda_0^*; b_0, G)} \right) + \frac{e^{-\rho T}}{\rho} (1 - \eta(C(\Lambda_0^*; 0, G) + G)) = 0. \quad (22)$$

As $C_{0,t}^* = C_0^*$ for all $t < T$ and $C_{0,t}^* = C_1^*$ for all $t \geq T$, the bond price satisfies

$$q_{0,t}^* = e^{-\rho t}, \text{ for } t < T; \quad q_{0,t}^* = e^{-\rho t} \frac{C_0^*}{C_1^*}, \text{ for } t \geq T. \quad (23)$$

Consequently, the competitive equilibrium interest rate is $r_{0,t}^* = \rho$ for both $t < T$ and

$t > T$. Immediately before $t = T$, the interest rate approaches ∞ , so $\{\tau_{0,t}^* - \rho\}_{t=0}^\infty$ is a Dirac delta function. We shall soon see how the Dirac delta interest rate at $t = T$ plays a key role.

In the context of this example, we now describe how Debortoli et al. use Lucas and Stokey's way of structuring a continuation debt term structure in order to motivate the continuation Ramsey planner to implement the continuation of the Ramsey plan. The Ramsey allocation (19) is piece-wise linear, so the Ramsey planner repurchases $\{b_{0,t}\}_{t=0}^\infty$ at time 0 and sells debt with the following coupon payment schedule: $\hat{b}_{0,t} = \hat{b}_0$ for all $t < T$ and $\hat{b}_{0,t} = \hat{b}_1$ for all $t \geq T$, making sure that \hat{b}_0 satisfies both the budget constraint (9) at time 0 and the budget constraint (10) at time $t = T$. We compute \hat{b}_0 and \hat{b}_1 as follows.

- To compute \hat{b}_1 , we use the bond price given in (23) and the property that the primary surplus under the Ramsey plan is constant for $t \geq 1$: $S(C_1^*) = \tau_1^* N_1^* - G = C_1^* (1 - \eta(C_1^* + G))$ to simplify the time- T budget equation (10) as follows:

$$\int_T^\infty \hat{b}_1 e^{-\rho t} \frac{C_0^*}{C_1^*} dt = \int_T^\infty S(C_1^*) e^{-\rho t} \frac{C_0^*}{C_1^*} dt.$$

This yields: $\hat{b}_1 = S(C_1^*)$. Simplifying the time 0 budget constraint (6) for the initial debt term structure $\{b_{0,t}\}$ we obtain

$$\frac{1 - e^{-\rho}}{\rho} (S(C_0^*) - b_0) + \frac{e^{-\rho}}{\rho} S(C_1^*) \frac{C_0^*}{C_1^*} = 0.$$

Combining the above two results gives $\hat{b}_1 = (e^{\rho T} - 1) \left(\frac{b_0}{C_0^*} - 1 + \eta(C_0^* + G) \right) C_1^*$.

- Using the preceding expression for \hat{b}_1 to rewrite the time-0 budget constraint (9) as follows:

$$\int_0^T e^{-\rho t} \hat{b}_0 dt + \int_T^\infty e^{-\rho t} \hat{b}_1 \frac{C_0^*}{C_1^*} dt = \int_0^T e^{-\rho t} S(C_0^*) dt + \int_T^\infty e^{-\rho t} S(C_1^*) \frac{C_0^*}{C_1^*} dt,$$

we obtain $\hat{b}_0 = C_0^* (1 - \eta(C_0^* + G))$.

In summary, we obtain the following Lucas-Stokey debt restructuring policy:

$$\hat{b}_{0,t} = \begin{cases} \hat{b}_0 = C_0^* (1 - \eta(C_0^* + G)), & t \in [0, T), \\ \hat{b}_1 = (e^{\rho T} - 1) \left(\frac{b_0}{C_0^*} - 1 + \eta(C_0^* + G) \right) C_1^*, & t \geq T. \end{cases} \quad (24)$$

We can state key findings of Debortoli, Nunes, and Yared (2021) in the context of our continuous-time version of their model.

Lemma 2. *When b_0 is not too high, debt structure (24) induces a time T continuation Ramsey planner to confirm the Ramsey plan. But for b_0 above a threshold b^* , debt structure (24) is unable to induce the continuation planner to confirm the Ramsey plan.*

Debortoli, Nunes, and Yared (2021)'s implementation resembles Alexander Hamilton's rescheduling of United States government debts at the beginning of the first Washington administration in 1790. Hamilton exchanged finite maturity Continental and state government bonds in exchange for bundles of US federal consols.⁵ Notice how in Figure 2 a Ramsey planner reschedules initial finite maturity debt streams with consols.

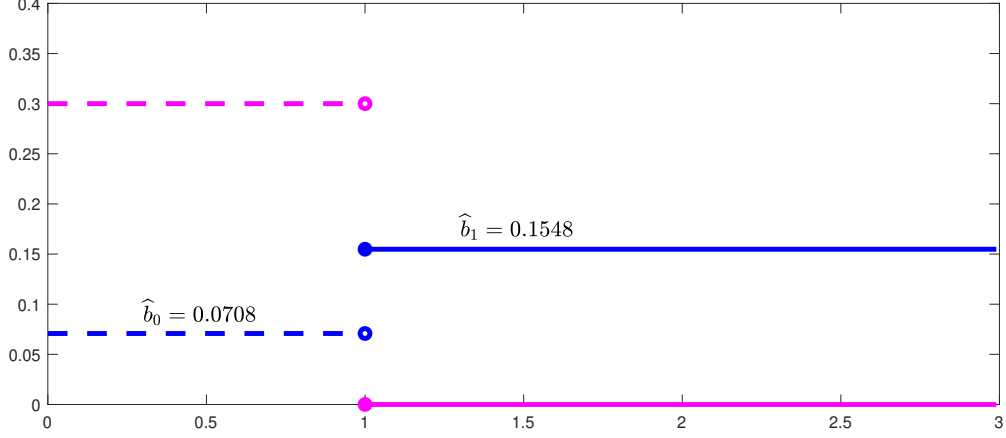
Numerical illustrations. To stay close to Debortoli et al.'s discrete time example, we set $T = 1$ and $(\gamma, \eta) = (1, 1)$, which makes $U(C, N) = \log C - N$. We use the Debortoli et al.'s one-period discount factor of 0.5 by setting $e^{-\rho} = 0.5$, so that $T = 1$ corresponds to 13.9 years with an annual discount rate of 5%. For these parameter values, it turns out that the threshold in Lemma 2 is $b^* = 0.35$. By setting $b_0 = .3$, we recover an example in which committing the government to finance Lucas and Stokey's recommended restructured debt term structure works. By setting $b_0 = .5$, we'll construct a version of Debortoli et al.'s counterexample in which it doesn't work.

Lucas-Stokey Implementation works when $b_0 = 0.3$. When $b_{0,t} = b_0 = 0.3$ for $t \in [0, 1)$ and $b_{0,t} = 0$ for $t \geq 1$, the Ramsey planner sets $C_{0,t}^* = C_0^* = 0.6987$ for $t \in [0, 1)$, $C_{0,t}^* = C_1^* = 0.4721$ for $t \geq 1$, and associated tax rates: $\tau_0^* = 0.3013$ and $\tau_1^* = 0.5273$. To induce the continuation Ramsey planner to confirm the Ramsey plan, the Ramsey planner repurchases $\{b_{0,t}\}_{t=0}^\infty$ and sells $\{\hat{b}_{0,t}\}_{t=0}^\infty$: $\hat{b}_{0,t} = \hat{b}_0 = 0.0708$ for $t \in [0, 1)$ and $\hat{b}_{0,t} = \hat{b}_1 = 0.1548$ for $t \geq 1$. This induces the continuation Ramsey planner at $t = 1$ to confirm the Ramsey plan: $C_{1,t}^* = C_1^* = 0.4721$ for all $t \geq 1$.

Lucas-Stokey Implementation doesn't work when $b_0 = 0.5$. When $b_{0,t} = b_0 = 0.5$ for $t \in [0, 1)$ and $b_{0,t} = 0$ for $t \geq 1$, the Ramsey planner sets $C_{0,t}^* = C_0^* = 0.7374$ for $t \in [0, 1)$ and $C_{0,t}^* = C_1^* = 0.1846$ for $t \geq 1$. To induce the continuation Ramsey planner to confirm the continuation of the Ramsey plan, Lucas and Stokey advise the Ramsey planner to repurchase $\{b_{0,t}\}_{t=0}^\infty$ and to construct a restructured debt term structure $\{\hat{b}_{0,t}\}_{t=0}^\infty$ in which $\hat{b}_{0,t} = \hat{b}_0 = 0.0462$ for $t \in [0, 1)$ and $\hat{b}_{0,t} = \hat{b}_1 = 0.1136$ for $t \geq 1$. To construct the associated

⁵See Hall and Sargent (2014).

(a) Initial debt term structure $b_{0,t}$ and restructured debt $\hat{b}_{0,t}$



(b) Ramsey plan and continuation Ramsey plan.

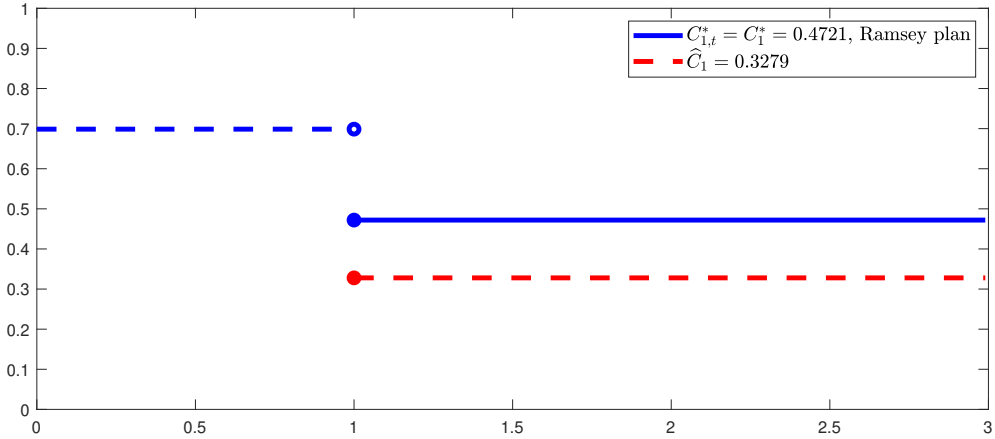


Figure 2: Implementation works when $b_0 = 0.3$. Ramsey plan: $C_0^* = 0.6987, C_1^* = 0.4721, \tau_0^* = 0.3013$, and $\tau_1^* = 0.5273$. Parameter values are: $G_t = 0.2, e^{-\rho} = 0.5$, and $\eta = 1$.

continuation Ramsey plan, first substitute the first order necessary condition $1 - \tau_{1,t} = C_{1,t}$ for labor into the primary surplus formula to get $S(C_{1,t}) = C_{1,t}(1 - (C_{1,t} + G))$. The implementation constraint for the continuation Ramsey planner is:

$$\int_1^{\infty} e^{-\rho(t-1)} \frac{\hat{b}_1}{C_{1,t}} dt \leq \int_1^{\infty} e^{-\rho(t-1)} \frac{S(C_{1,t})}{C_{1,t}} dt.$$

Form a Lagrangian for the continuation Ramsey planner to obtain $C_{1,t} = C_{1,1}$ and $S(C_{1,t}) = \hat{b}_1$ for all $t \geq 1$. Solve $S(C_{1,1}) = \hat{b}_1$ for $C_{1,1}$ to obtain two roots: $C_{1,1} = C_1^* = 0.1846$,

which describes the continuation of the Ramsey plan, and $C_{1,1} = \hat{C}_1 = 0.6154$, which describes the continuation Ramsey plan. To verify that the continuation Ramsey planner sets $C_{1,1} = \hat{C}_1 = 0.6154$ in order to maximize its objective function, form the Lagrangian:

$$\mathcal{L}_1 = \int_1^\infty e^{-\rho(t-1)} \left[\log C_{1,t} - (C_{1,t} + G) + \Lambda_1 \left(1 - (C_{1,t} + G) - \frac{\hat{b}_1}{C_{1,t}} \right) \right] dt.$$

We can show that

$$U(\hat{C}_1, \hat{C}_1 + G) > U(C_1^*, C_1^* + G)$$

by using $\hat{C}_1 = 0.6154$, $C_1^* = 0.1846$, and

$$-1.8769 = \int_1^\infty e^{-\rho(t-1)} U(\hat{C}_1, \hat{C}_1 + G) dt > \int_1^\infty e^{-\rho(t-1)} U(C_1^*, C_1^* + G) dt = -2.9926.$$

This completes our verification of Debortoli et al.’s counterexample in which a Ramsey plan can’t be implemented by restructuring a Càdlàg $\{b_{0,t}; t \geq 0\}$ debt structure.

It is not possible to find a counterexample to a counterexample, so we won’t try. Instead, we’ll seek an arrangement with minimal additional obligations that continuation Ramsey planners are bound to honor, ones that, in our context, are faithful to Lucas and Stokey’s intention to give the government access to complete markets.

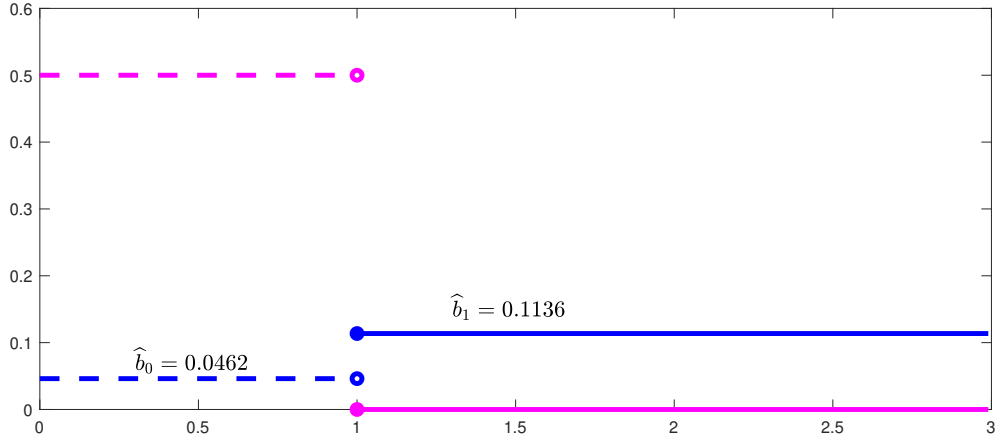
5 Expanding the Contractible Subspace

We implement a Ramsey plan by adding instantaneous debt B_t to the *contractible subspace*.⁶ In addition to a rescheduled debt structure $\{\hat{b}_{0,t}, t \geq T\}$, the continuation Ramsey planner must also service a rescheduled instantaneous debt $\{\hat{B}_t, t \geq T\}$. Continuation Ramsey planners are also obligated to protect the ‘purchasing power’ of the rescheduled instantaneous debt balance \hat{B}_t over the next instant: $\hat{B}_t U_{C,t}$. Because we impose no restriction on the “purchasing power” of term debt $\{\hat{b}_{0,t}, t \geq T\}$, we dub this purchasing power obligation a “local commitment.” This extension of the contractible subspace allows the Ramsey planner to implement the Ramsey plan.

The remainder of this section shows how this extension allows the Ramsey planner to implement the Ramsey plan. In subsection 5.1, we show why our implementation of Ram-

⁶This is the same type of short-term debt widely used in asset pricing, e.g., Black and Scholes (1973), Merton (1971), and Harrison and Kreps (1979).

(a) Initial debt term structure $b_{0,t}$ and restructured debt $\hat{b}_{0,t}$



(b) Ramsey plan and continuation Ramsey plan.

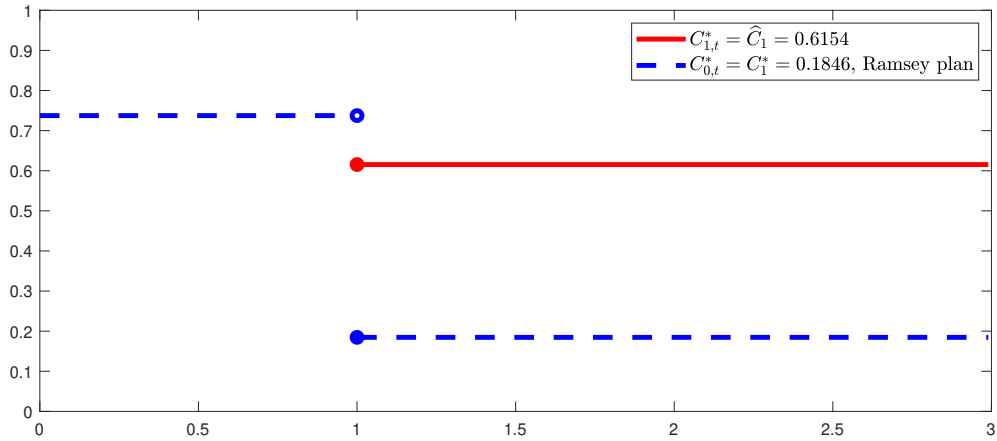


Figure 3: Implementation doesn't work when $b_0 = 0.5$. Ramsey plan: $C_0^* = 0.7374$, $C_1^* = 0.1846$, $\tau_0^* = 0.2626$, and $\tau_1^* = 0.8154$. Continuation Ramsey plan: $\hat{C}_1 = 0.6154$ and $\hat{\tau}_1 = 0.3846$. Parameter values are: $G_t = 0.2$, $e^{-\rho} = 0.5$, and $\eta = 1$.

sey plan works in the context of Debortoli et al.'s counterexample where $b_0 = 0.5$. In the same context, in subsection 5.2, we show that our implementation and Lucas and Stokey' implementation generate different prescriptions for the government's gross of interest surplus and other equilibrium objects and tell how our implementation captures the Barro (1979) prescription that the government should use gross of interest deficits to smooth tax distortions.

5.1 Implementing Ramsey Plan with Instantaneous Debt

We focus on the setting of Debortoli et al.'s counterexample in which $b_0 = 0.5$ and $T = 1$. Using $q_{0,t}^* = e^{-\rho t}$ and the formula for $S(C_0^*)$ given in (18) for $t < 1$, and then applying (15) to $t = 1-$, we obtain:

$$\Pi_{1-}^* = \frac{1}{e^{-\rho}} \int_0^{1-} e^{-\rho s} (b_0 - S(C_0^*)) ds = \frac{(e^\rho - 1)}{\rho} \left(\frac{b_0}{C_0^*} - 1 + (C_0^* + G) \right) C_0^*. \quad (25)$$

Next, using $q_{0,1-}^*/q_{0,1}^* = C_1^*/C_0^*$, (15), and (25), we obtain:

$$\Pi_1^* = \frac{1}{q_{0,1}^*} \int_0^1 q_{0,s}^* (b_{0,s} - S(C_{0,s}^*)) ds = \frac{q_{0,1-}^*}{q_{0,1}^*} \frac{1}{q_{0,1-}^*} \int_0^1 q_{0,s}^* (b_{0,s} - S(C_{0,s}^*)) ds = \frac{C_1^*}{C_0^*} \Pi_{1-}^*. \quad (26)$$

Combining (25) and (26) we deduce

$$\frac{\Pi_{1-}^*}{C_0^*} = \frac{\Pi_1^*}{C_1^*} = \frac{(e^\rho - 1)}{\rho} \left(\frac{b_0}{C_0^*} - 1 + (C_0^* + G) \right), \quad (27)$$

which requires that a continuation Ramsey planner choose to smooth the representative household's "purchasing power" over time, including at $t = 1$.

We can restrict debt restructuring policies to a set in which the Ramsey planner leaves the initial debt term structure $\{b_{0,t}\}_{t=0}^\infty$ untouched and instead uses instantaneous debt to finance accumulated deficits Π_t^* . We propose the following debt restructuring policy.

Debt Restructure 1. *Refinance $\{G_t, b_{0,t}\}_{t=0}^\infty$ as follows:*

- Set $\{\hat{b}_{0,t}\}_{t=0}^\infty = \{b_{0,t}\}_{t=0}^\infty$.
- Start with $B_0 = 0$, accumulate instantaneous debt balance B_t according to $dB_t = d\Pi_t^*$ for $t \in [0, 1)$; restructure instantaneous debt from B_{1-} to $\hat{B}_1 = \frac{C_1^*}{C_0^*} B_{1-}$ and leave this new balance to time-1 government.

Why do we require the Ramsey planner to restructure B_{1-} to $\hat{B}_1 = \frac{C_1^*}{C_0^*} B_{1-}$ at $t = 1-$? To answer this question, we begin by noting that under this restructuring, $B_{1-} = \Pi_{1-}^*$ and $\hat{B}_1 = \frac{C_1^*}{C_0^*} B_{1-} = \frac{C_1^*}{C_0^*} \Pi_{1-}^* = \Pi_1^*$, where the last equality follows from (27). Therefore, this policy tracks the evolution of $\{\Pi_t^*; t \geq 0\}$ perfectly. This suggests that it might be possible to implement a Ramsey plan without rescheduling the initial term debt structure.

To proceed, notice that, since $U_C(C_{0,1}^*, C_{0,1}^* + G)\hat{B}_1 = U_C(C_{0,1-}^*, C_{0,1-}^* + G)B_{1-}$, the Ramsey plan preserves the "purchasing power" of instantaneous debt. A key to inducing

a continuation government to confirm the Ramsey plan is to require that it preserves the “purchasing power” of instantaneous debt, which requires that

$$U_C(C_{1,1}, C_{1,1} + G)\widehat{B}_1 = U_C(C_{0,1-}, C_{0,1-} + G)B_{1-}. \quad (28)$$

where $C_{1,1}$ is chosen by the continuation Ramsey planner at $t = 1$. We must verify that a continuation Ramsey planner that confronts \widehat{B}_1 wants to set $C_{1,1} = C_1^*$. Substituting $C_{0,1-} = C_0^*$ into (28), we obtain

$$\frac{\widehat{B}_1}{C_{1,1}} = \frac{B_{1-}}{C_0^*}. \quad (29)$$

Substituting $B_{1-} = \Pi_{1-}^*$ and $\widehat{B}_1 = \Pi_1^*$ into (29) and using (27), we obtain $C_{1,1} = C_1^*$, which is the continuation of the Ramsey plan.

Facing \widehat{B}_1 , the continuation Ramsey planner chooses $\{C_{1,t}; t > 1\}$ to maximize the household’s utility subject to the following continuation implementability constraint:

$$\frac{\widehat{B}_1}{C_{1,1}} \leq \int_1^\infty e^{-\rho(t-1)} \frac{S(C_{1,t}) - 0}{C_{1,t}} dt.$$

After forming and extremizing a Lagrangian for the continuation Ramsey planner, we can conclude that: 1) $C_{1,t}$ is constant for all $t > 1$; 2) Continuation IC and local commitment constraint (29) imply

$$\frac{1}{\rho} \frac{S(C_{1,t}) - 0}{C_{1,t}} = \frac{\widehat{B}_1}{C_{1,1}} = \frac{B_{1-}}{C_0^*} = \frac{1}{\rho} \frac{S(C_1^*) - 0}{C_1^*}, \quad \forall t > 1;$$

and 3) the continuation Ramsey planner chooses $C_{1,t} = C_1^*$ for all $t \geq 1$, thus confirming the tail of the Ramsey plan.

For the special $b_0 = .5$ setting that leads to Debortoli et al.’s counterexample, $\widehat{B}_1 = \Pi_1^* = 0.1639$ and $C_{1,t} = C_1^* = 0.1846$ for $t \geq 1$, which confirms the Ramsey plan. In Figure 4, we plot the equilibrium bond price $q_{0,t}^*$ and the interest rate $r_{0,t}^*$. For $t \in [0, 1)$, $q_{0,t}^* = e^{-\rho t} = 0.5^t$ as $r_{0,t}^* = \rho = -\ln(0.5)$. Importantly, note that the bond price jumps from $q_{0,1-}^* = e^{-\rho} = 0.5$ to $q_{0,1}^* = 0.5 \frac{C_0^*}{C_1^*} = 2.0$ as the equilibrium interest rate at $t = 1$ approaches ∞ . Technically, $r_{0,t}^*$ is a Dirac delta function that makes $q_{0,t}^*$ jump at $t = 1$, as we discussed in section 4. This jump is optimal because Ramsey planner wants to smooth the ‘purchasing power’ of the instantaneous debt at all t including $t = 1$ where $b_{0,t}$ discretely jumps. Finally, for $t \in [1, \infty)$, $q_{0,t}^* = q_{0,1}^* e^{-\rho t} = 2.0 \times 0.5^t$ as $r_{0,t}^* = \rho = -\ln(0.5)$ for $t > 1$.

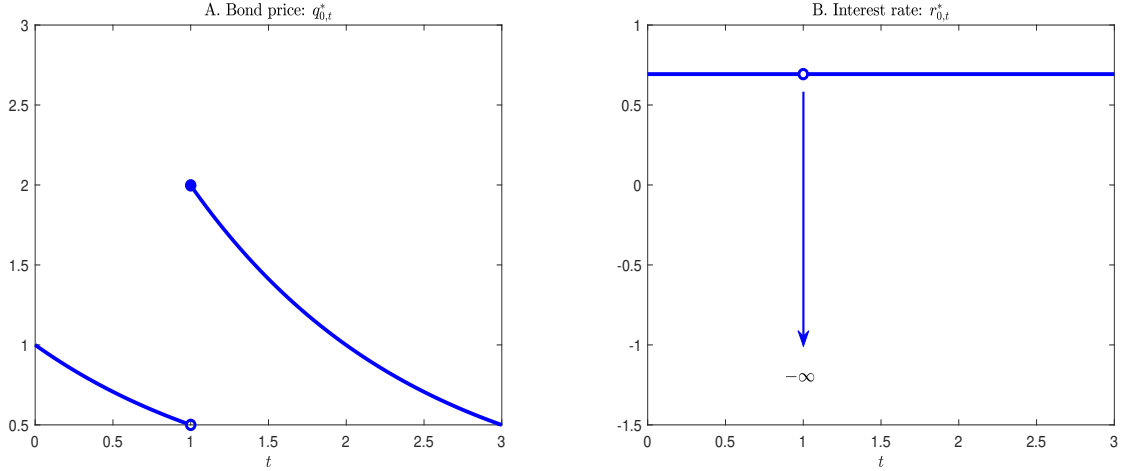


Figure 4: **Bond price $q_{0,t}^*$ and interest rate $r_{0,t}^*$ under Ramsey plan when $b_0 = 0.5$.** The equilibrium bond price process is: $q_{0,t}^* = e^{-\rho t}$ for $t \in [0, 1)$ and $q_{0,t}^* = q_{0,1}^* e^{-\rho(t-1)}$ for $t > 1$; it jumps to $q_{0,1}^* = 2.0$ at $t = 1$ from $q_{0,1-}^* = 0.5$. The equilibrium interest rate is proportional to a Dirac delta function: $r_{0,t}^* = \rho$ for $t \in [0, 1)$ and $t > 1$, and $r_{0,t}^* \rightarrow -\infty$ at $t = 1$. Parameter values are: $G = 0.2$, $e^{-\rho} = 0.5$, and $\eta = 1$.

In Debortoli et al.’s counterexample, restructuring the initial term debt structure could not implement the Ramsey plan because the restructured term debt stream $\{\hat{b}_{0,t}; t \geq 1\}$ proposed by Lucas and Stokey has long duration, presenting continuation governments incentives to dilute its “purchasing power” by setting $C_{1,t} = \hat{C}_1 = 0.6154$ (and $\tau_{1,t} = \hat{\tau}_1 = 0.3846$) for $t \geq 1$, instead of $C_{1,t} = C_1^* = 0.1846$ and $\tau_1^* = 0.8154$, as called for in the Ramsey plan.

Availability of instantaneous debt and the local commitment constraint (29) let the Ramsey planner implement its plan by using restructuring policy 1. The instantaneous debt balance \hat{B}_1 includes information about both the price (via $r_{0,t} = \rho$) and the quantity (\hat{b}_1) of the government’s liability: $\hat{B}_1 = \int_1^\infty e^{-\rho(t-1)} \hat{b}_1 dt = \hat{b}_1 / \rho$. The (local) commitment condition (29) prohibits time-1 continuation planner from diluting the household’s “purchasing power” of \hat{B}_1 , which is something that the continuation Ramsey planner wants to do when it confronts the Càdlàg $\{\hat{b}_{0,t}; t \geq 0\}$ debt structure without instantaneous debt in Debortoli et al.’s counterexample.

Unlike Lucas and Stokey’s, in our implementation the government does not restructure the initial term debt. Instead it books the cumulative liability Π^* and services it by issuing instantaneous debt balance $B_t = \Pi_t^*$, something that it is always feasible for the continuation Ramsey planner to do. To implement the Ramsey plan, it is sufficient to require

that the continuation Ramsey planner preserves the purchasing power of this instantaneous debt.

In Lucas and Stokey’s implementation, there is no instantaneous debt, so their Ramsey planner has fewer tools than ours. Their term debt restructuring (from $\{b_{0,t}\}$ to $\{\hat{b}_{0,t}\}$) has to satisfy two restrictions that in Debortoli et al.’s counterexample can be incompatible: satisfying the government’s budget constraint and inducing the continuation Ramsey planner to confirm a tax plan. These two restrictions are compatible when $b_0 = .3$ in Debortoli et al.’s model, but not when $b_0 = 0.5$.

5.2 Different Implementations Imply Different Gross of Interest Surplus

Given an equilibrium, we define the gross of interest surplus process as follows (Ljungqvist and Sargent, 2018):

$$\Theta_{0,t} = S_{0,t} - \underbrace{(b_{0,t} + r_{0,t}B_t)}_{\text{total interest payments}} \quad , \quad (30)$$

where $S_{0,t} = \tau_{0,t}N_{0,t} - G_t$ is the primary surplus and the second term includes both the term debt and instantaneous debt interest payments.

Next, we show that even in settings where both our implementation and Lucas and Stokey’s implementation work, they imply different gross of interest surpluses. This is because our implementation does not touch the initial term debt $b_{0,t}$ and uses instantaneous debt balance B_t to track cumulative liabilities, while Lucas and Stokey’s implementation constantly restructures term debt and does not use instantaneous debt.

In Figure 5, we show that in our implementation the instantaneous debt balance $B_t = \Pi_t^*$ increases exponentially over time until $t = 1$, then decreases discontinuously from 0.33 to 0.22 at $t = 1$ in order to preserve the ‘purchasing power’ of instantaneous debt balance B_t , and then stays constant at 0.22 for $t > 1$ (panel A). In contrast, by default, $B_t = 0$ for all t in Lucas and Stokey’s implementation. This difference together with no term debt restructuring in our implementation and $\hat{b}_0 = 0.0708, \hat{b}_1 = 1548$ in Lucas and Stokey’s implementation imply a very different gross of interest surplus dynamics (panel B). Because $S_{0,t}^* = \hat{b}_{0,t}$ and $B_{0,t} = 0$ in Lucas and Stokey’s implementation, the government runs no gross of interest surplus at any t in their implementation: $\Theta_{0,t} = S_{0,t}^* - \hat{b}_{0,t} = 0$. In contrast, in our implementation, the government first runs a gross of interest deficit ($\Theta_{0,t} = S_{0,t}^* - b_{0,t} - r_{0,t}^*B_t < 0$) until for all $t < 1$ and then zero gross of interest deficit for

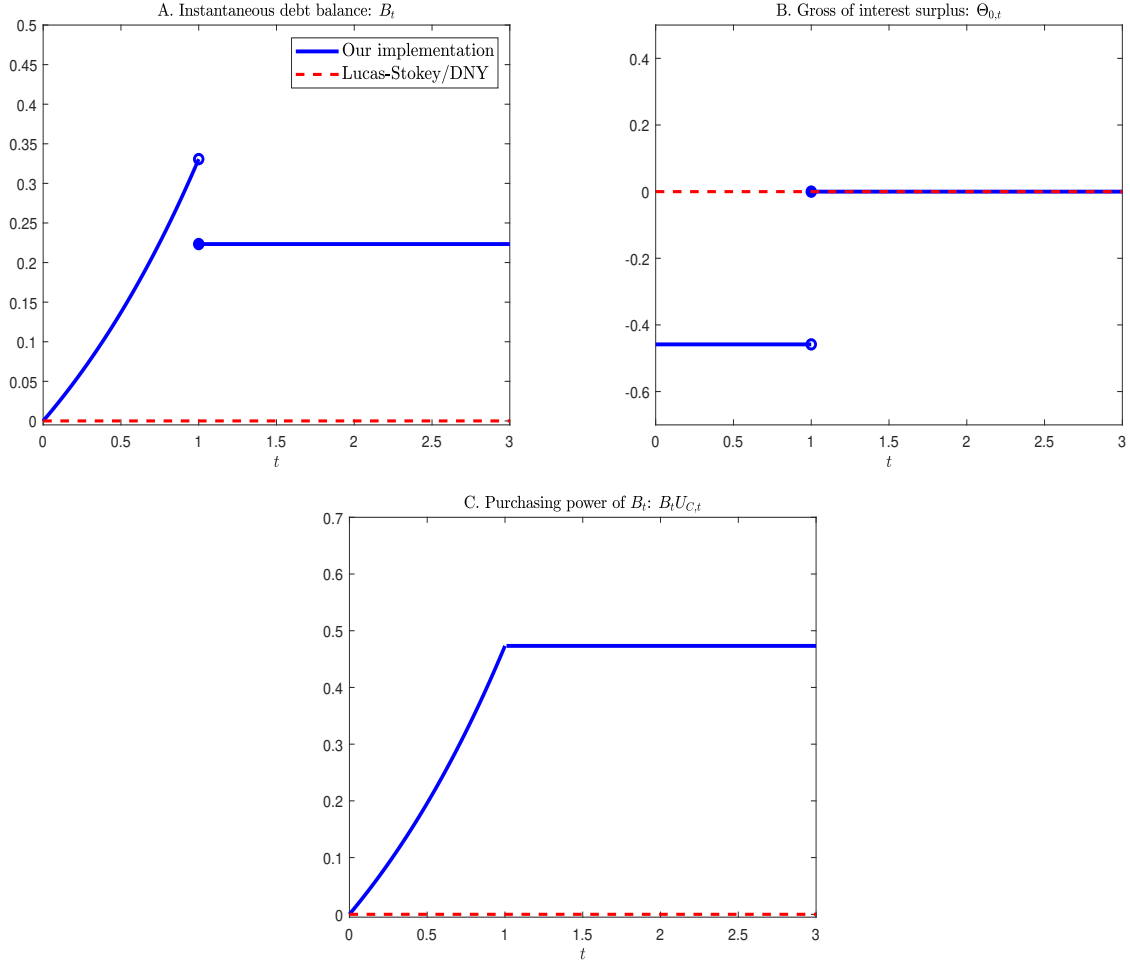


Figure 5: Instantaneous debt balance B_t , gross of interest surplus $\Theta_{0,t}$, and ‘purchasing power’ of instantaneous debt $B_t U_{C,t}$ under Ramsey plan when $b_0 = 0.3$. Parameter values: $G_t = 0.2$, $e^{-\rho} = 0.5$, and $\eta = 1$.

all $t \geq 1$. This is because absent term debt restructuring, the government debt obligations ($b_{0,t} = 0.3$ for $t < 1$ and zero afterwards) have to be booked via short term debt balance in our implementation. Finally, panel C highlights the key mechanism of our implementation: preserving the ‘purchasing power’ of instantaneous debt $B_t U_{C,t}$ under the local commitment condition (by making it continuous in t) facilitates the implementation of Ramsey plan. Note that $B_t U_{C,t}$ is continuous in t even when B_t is discontinuous.

6 A Dynamic Program

This section presents a recursive representation of a Ramsey plan. First, we introduce a state variable $\{X_t; t \geq 0\}$. Second, we formulate a dynamic programming problem for a given X_0 . Third we characterize the Ramsey plan.

6.1 Introducing State Variable X_t

For debt service flow $\{b_{0,s}, s \geq 0\}$, we define:

$$\Pi_t = \int_0^t \frac{q_{0,s}}{q_{0,t}} (b_{0,s} + G_s - \tau_{0,s}N_{0,s}) ds, \quad \Pi_0 = 0. \quad (31)$$

Here $(b_{0,s} + G_s - \tau_{0,s}N_{0,s}) ds$ is the (flow) increase of the government's liability over a small time interval ds without touching term debt at all and the ratio $q_{0,s}/q_{0,t}$ compounds this flow's contribution to the government's time- t liability stock.⁷ Since we assume that $\{G_t\}$ and $\{b_{0,t}\}$ are Càdlàg processes, optimal allocations $\{C_{0,t}, N_{0,t}; t \geq 0\}$ may not be continuous. Let $\mathcal{T} = \{0 < t_1 < t_2 < \dots\}$ denote a set of countable (possibly infinite) points where the $\{C_{0,t}; t \geq 0\}$ process jumps.

For $t \notin \mathcal{T}$ where $\{\Pi_t\}$ does not jump, $q_{0,t}$ and Π_t evolve continuously: $dq_{0,t} = -r_{0,t}q_{0,t}dt$ and

$$d\Pi_t = r_{0,t}\Pi_t dt + (b_{0,t} + G_t - \tau_{0,t}N_{0,t}) dt. \quad (32)$$

Where $\{\Pi_t\}$ jumps at $t \in \mathcal{T}$, we have

$$\Pi_t = \Pi_{t-} \frac{q_{0,t-}}{q_{0,t}}, \quad t \in \mathcal{T}. \quad (33)$$

Thus, the time-0 value of the government's time t cumulative liability $q_{0,t}\Pi_t$ is the same before and after a jump.

Define

$$X_t = \Pi_t U_{C,t}, \quad (34)$$

which will serve as a state variable for a recursive formulation of the Ramsey problem. Evidently, X_t incorporates both information about the household's wealth Π_t and its marginal

⁷When the interest rate is a constant r over (s, t) , then $\frac{q_{0,s}}{q_{0,t}} = e^{r(t-s)} > 1$.

utility of consumption $U_{C,t}$. For $t \notin \mathcal{T}$ where $\{\Pi_t\}$ does not jump, X_t is continuous because $U_{C,t}$ is also continuous. Differentiating X_t both sides of (34) with respect to t and using (32) together with (4) and (5), we obtain:

$$\begin{aligned}
dX_t &= U_{C,t}d\Pi_t + \Pi_t d(e^{\rho t} q_{0,t} U_{C,0}) \\
&= U_{C,t} \left[-\Pi_t \frac{dq_{0,t}}{q_{0,t}} + (b_{0,t} + G_t - \tau_{0,t} N_{0,t}) dt \right] + \Pi_t \left(q_{0,t} U_{C,0} \rho e^{\rho t} dt + e^{\rho t} U_{C,0} q_{0,t} \frac{dq_{0,t}}{q_{0,t}} \right) \\
&= -X_t \frac{dq_{0,t}}{q_{0,t}} + (b_{0,t} U_{C,t} - C_{0,t} U_{C,t} - N_{0,t} U_{N,t}) dt + \rho X_t dt + X_t \frac{dq_{0,t}}{q_{0,t}} \\
&= (\rho X_t - C_{0,t} U_{C,t} - N_{0,t} U_{N,t} + b_{0,t} U_{C,t}) dt.
\end{aligned} \tag{35}$$

What happens where $\{\Pi_t\}$ jumps at $t \in \mathcal{T}$? Using (33) and first-order necessary condition (5), we obtain $X_{t-} = \Pi_{t-} U_{C,t-} = \Pi_{t-} q_{0,t-} e^{\rho t-} U_{C,0} = \Pi_t q_{0,t} e^{\rho t} U_{C,0} = \Pi_t U_{C,t} = X_t$. We have established:

Lemma 3. X_t defined in (34) is continuous in time. It evolves according to (35).

Because welfare maximization calls for continuity of the household's net worth Π_t multiplied by the marginal utility $U_{C,t}$, the Ramsey planner wants to make X_t continuous, even where the initial debt structure $\{b_{0,t}\}$ or government spending process $\{G_t\}$ jumps.

Since the initial debt structure $b_{0,t}$ and government spending G_t are generally time dependent, the Ramsey problem is time inhomogeneous, so t will appear in Ramsey planner's value function. Consequently, we reformulate the Ramsey problem in the following may two steps. First, for a given X_0 , we define and solve a dynamic programming (DP) problem with X_t and time t being the state variables in subsection 6.2. In subsection 6.3 we use the DP problem to characterize a Ramsey plan.

6.2 Formulating a Dynamic Program (DP)

Definition 3 (DP Problem). *Confronting X_0 at $t = 0$, a government solves*

$$\max_{\{C_{0,s}; s \geq 0\}} \int_0^\infty e^{-\rho s} U(C_{0,s}, C_{0,s} + G_s) ds, \tag{36}$$

where maximization is subject to law of motion evolution (35).

This can be formulated as a dynamic programming problem (see e.g., Fleming and Soner, 2006). Let $V(X_0, 0)$ denote the (optimal) value function for this problem. To solve

$V(X_0, 0)$, we first construct the following time- t optimization problem:

$$\max_{\{C_{0,s}; s \geq t\}} \int_t^\infty e^{-\rho(s-t)} U(C_{0,s}, C_{0,s} + G_s) ds. \quad (37)$$

subject to (35). Let $V(X_t, t)$ denote the (optimal) value function for (37). For all $t \geq 0$, the value function $V(X_t, t)$ defined in (37) satisfies the following HJB equation:

$$\rho V = \frac{\partial V}{\partial t} + \max_{C_{0,t}} U(C_{0,t}, C_{0,t} + G_t) + \frac{\partial V}{\partial X_t} (\rho X_t - C_{0,t} U_{C,t} - (C_{0,t} + G_t) U_{N,t} + b_{0,t} U_{C,t}). \quad (38)$$

We guess and verify that $V(X_t, t)$ is additively separable in X_t and t :

$$V(X_t, t) = -\Phi_0 X_t + H(t; \Phi_0); \quad t \geq 0, \quad (39)$$

where Φ_0 is an object to be chosen by the Ramsey planner. Substituting (39) into (38), we obtain the following differential equation for all $t \geq 0$:

$$\rho H(t; \Phi_0) = H'(t; \Phi_0) + \max_{C_{0,t}} U(C_{0,t}, C_{0,t} + G_t) + \Phi_0 (C_{0,t} U_{C,t} + (C_{0,t} + G_t) U_{N,t} - b_{0,t} U_{C,t}). \quad (40)$$

Let $C(\Phi_0; G_t, b_{0,t})$ denote the function for $C_{0,t}$ that maximizes (40):

$$C(\Phi_0; G_t, b_{0,t}) = \arg \max_{C_{0,t}} U(C_{0,t}, C_{0,t} + G_t) + \Phi_0 (C_{0,t} U_{C,t} + (C_{0,t} + G_t) U_{N,t} - b_{0,t} U_{C,t}). \quad (41)$$

Proposition 3. *Choices of $\{C_{0,t}\}$ for a DP Problem indexed by $\Phi_0 \geq 0$ satisfy (41) and attain the value function*

$$V(X_t, t; \Phi_0) = -\Phi_0 X_t + H(t; \Phi_0), \quad (42)$$

where $H(t; \Phi_0)$ is given by

$$H(t; \Phi_0) = \int_t^\infty e^{-\rho(s-t)} \left[U(C(\Phi_0; b_{0,s}, G_s), C(\Phi_0; b_{0,s}, G_s) + G_s) + \Phi_0 (C(\Phi_0; b_{0,s}, G_s) U_{C,s} + (C(\Phi_0; b_{0,s}, G_s) + G_s) U_{N,s} - b_{0,s} U_{C,s}) \right] ds. \quad (43)$$

and $N(\Phi_0, G_t, b_{0,t}) = C(\Phi_0, G_t, b_{0,t}) + G_t$.

We next turn to how the Ramsey planner chooses the scalar Φ_0 that pins down the value function for the **DP Problem**.

6.3 Characterizing the Ramsey Plan

The consumption stream $\{C_{0,t}\}$ is a Càdlàg process: $C_{0,0} = \lim_{s \downarrow 0} C_{0,s} = C(\Phi_0; b_{0,0}, G_0)$, which implies that $C_{0,0}$ is a function of Φ_0 . Subject to implementability constraint (7), the Ramsey planner chooses Φ_0 , or equivalently consumption $C_{0,0}$ according to

$$W = \max_{\Phi_0} V(X_0, 0; \Phi_0), \quad (44)$$

where $V(X_0, 0; \Phi_0)$ is given in (39). When solving the problem on the right side of (44), the Ramsey planner takes as given the initial debt term structure $\{b_{0,t}; t \geq 0\}$ and $X_0 = 0$ (implied by $\Pi_0 = 0$). The implementability constraint requires that Φ_0 satisfies $I_0(\Phi) = 0$, where

$$I_0(\Phi_0) = \int_0^\infty e^{-\rho t} (C(\Phi_0, b_{0,t}, G_t)U_{C,t} + (C(\Phi_0, b_{0,t}, G_t) + G_t)U_{N,t} - b_{0,t}U_{C,t}) dt. \quad (45)$$

Let $\mathcal{A} = \{\Phi_0 \geq 0 : I_0(\Phi_0) = 0\}$ denote the admissible set for Φ_0 . Using (42), the Ramsey s problem is

$$W = \max_{\Phi_0 \in \mathcal{A}} V(0, 0) = \max_{\Phi_0 \in \mathcal{A}} \int_0^\infty e^{-\rho t} U(C(\Phi_0, b_{0,t}, G_t), C(\Phi_0, b_{0,t}, G_t) + G_t) dt.$$

Proposition 4. *When it exists, a Ramsey plan satisfies*

$$\begin{aligned} C_{0,t}^* &= C(\Phi_0^*, b_{0,t}, G_t); & N_{0,t}^* &= C_{0,t}^* + G_t; \\ \tau_{0,t}^* &= 1 + \frac{U_{N,t}(C_{0,t}^*, N_{0,t}^*)}{U_{C,t}(C_{0,t}^*, N_{0,t}^*)}; & q_{0,t}^* &= e^{-\rho t} \frac{U_{C,t}(C_{0,t}^*, N_{0,t}^*)}{U_{C,0}(C_{0,0}^*, N_{0,0}^*)} \end{aligned}$$

where $\Phi_0^* = \arg \max_{\Phi_0 \in \mathcal{A}} V(0, 0)$. Under a Ramsey plan, $X_t = X_t^*$ where

$$X_t^* = \int_t^\infty e^{-\rho(s-t)} [C_{0,s}^* U_{C,t}(C_{0,t}^*, N_{0,t}^*) + N_{0,s}^* U_{N,t}(C_{0,t}^*, N_{0,t}^*) - b_{0,s} U_{C,t}(C_{0,t}^*, N_{0,t}^*)] ds \quad (46)$$

and the household's value is

$$W = \int_0^{\infty} e^{-\rho t} U(C_{0,t}^*, C_{0,t}^* + G_t) dt. \quad (47)$$

Proposition 5. *A Ramsey plan computed with the Proposition 2 Lagrangian method equals the Ramsey plan computed with the Proposition 4 Bellman equation.*

7 Ramsey Plans for Continuous $\{b_{0,t}, G_t\}$ Processes

This section constructs Ramsey plans for some settings with continuous $\{b_{0,t}, G_t; t \geq 0\}$ processes and uses them to illustrate salient outcomes. In addition to illustrating forces that shape tax rates and interest rates, studying economies with continuous $\{b_{0,t}, G_t; t \geq 0\}$ processes provides techniques that will help us understand outcomes with Càdlàg $\{b_{0,t}, G_t; t \geq 0\}$ processes, as we show in section 9. It also facilitates our comparison with discrete-time models.

Assumption 2. *Exogenous flows of government expenditures $\vec{G}_0 = \{G_t; t \geq 0\}$ and debt-service payouts $\vec{b}_0 = \{b_{0,t}; t \geq 0\}$ are continuous functions of time.*

Under Assumption 2, the debt price process $\{q_{0,t}^*; t \geq 0\}$ and cumulative liability process $\{\Pi_t^*; t \geq 0\}$ are both continuous in time. We later show that the continuity property of $\{\Pi_t^*; t \geq 0\}$ substantially simplifies our exposition of the implementation result. The following lemma describes the Ramsey plan when $\{b_{0,t}, G_t\}$ is continuous.

Lemma 4. *Under Assumption 2, the Ramsey plan characterized in Proposition 4 is continuous in time, i.e., the consumption and labor supply processes $\{C_{0,t}^* = C(\Phi_0^*; b_{0,t}, G_t), N_{0,t}^* = N(\Phi_0^*; b_{0,t}, G_t)\}$, tax rate process $\{\tau_{0,t}^*; t \geq 0\}$, instantaneous interest rate process $\{r_{0,t}^*; t \geq 0\}$, and cumulative liabilities Π_t^* process are all continuous in time.*

Lemma 4 follows immediately by applying Proposition 4 under the additional Assumption 2. Next, we analyze several example economies in which both $\{b_{0,t}\}$ and $\{g_t\}$ are continuous in time.

7.1 $\{b_{0,t}\}$ is Exponential

Consider an exponential debt term structure: $b_{0,t} = b_0 e^{-\theta t}$, where $b_0 = 0.2$ and $\theta = 0.6$ and $G_t = 0.2$ for all t . Figure 6 shows the Ramsey solution and its implications. Under the plan,

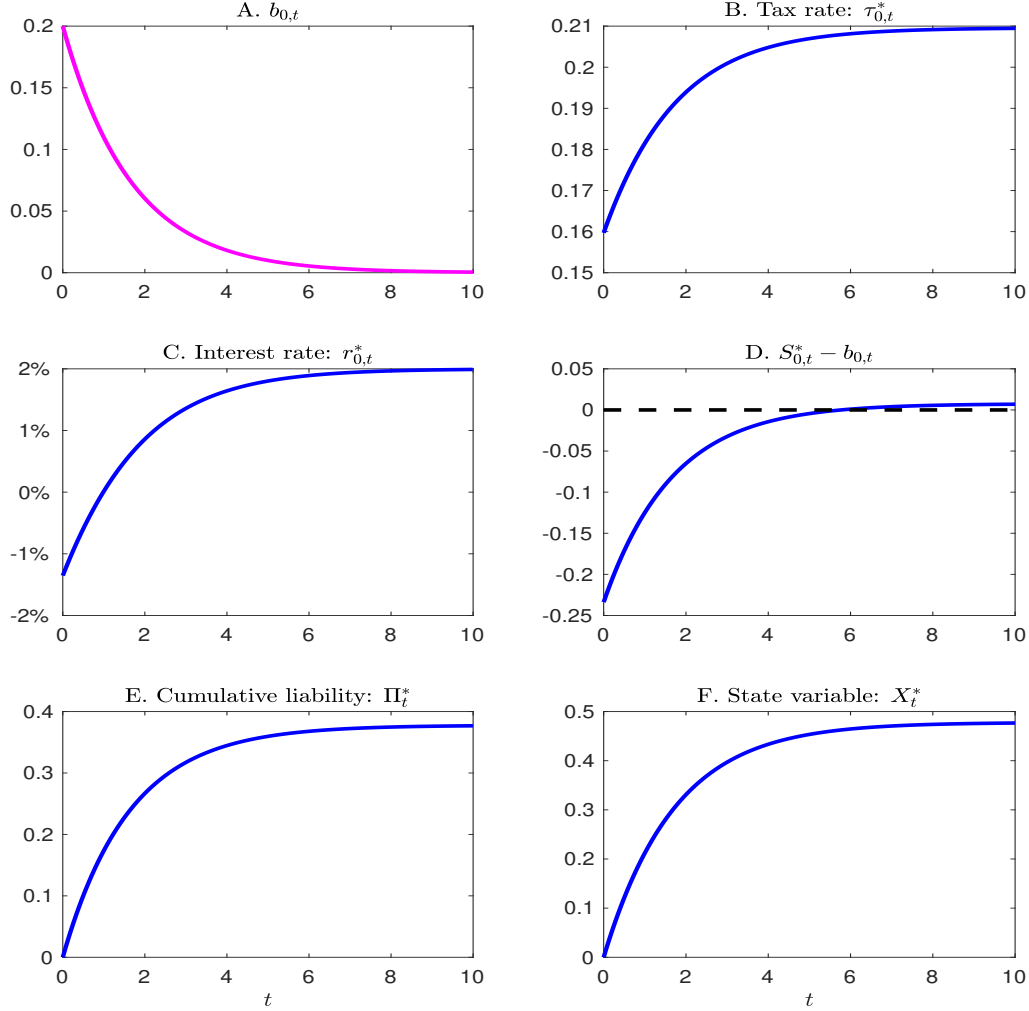


Figure 6: Ramsey plan for exponentially decaying $b_{0,t} = b_0 e^{-\theta t}$. Parameter values: $b_0 = 0.2$, $\theta = 0.6$; $G_t = 0.2$, $\rho = 0.02$, and $\eta = 1$.

the tax rate $\tau_{0,t}^*$ increases over time (panel B), which implies that consumption $C_{0,t}^* = 1 - \tau_{0,t}^*$ and hence labor supply $N_{0,t}^* = C_{0,t}^* + G_t$ decreases over time. The equilibrium interest rate is $r_{0,t}^* = \rho + g_{0,t}^*$ where $g_{0,t}^* = \dot{C}_{0,t}^*/C_{0,t}^*$ is the equilibrium consumption growth rate. Note that the consumption growth rate $g_{0,t}^*$ is always negative:⁸ lower than the (minus) household's

⁸In this example, $C_{0,t}^* = 2b_0 e^{-\theta t} \Phi_0^* / (\sqrt{1 + 4b_0 e^{-\theta t} \Phi_0^* (1 + \Phi_0^*)} - 1)$ and $\Phi_0^* = 0.2652$. Therefore, the consumption growth rate is negative: $g_{0,t}^* = dC_{0,t}^*/dt / C_{0,t}^* < 0$ because $dC_{0,t}^*/dt = -b_0 \Phi_0^* \theta e^{-\theta t} / \sqrt{1 + 4b_0 e^{-\theta t} \Phi_0^* (1 + \Phi_0^*)} < 0$.

discount rate $(-\rho)$ at $t = 0$, turning less negative over time and eventually approaches zero. As a result, the interest rate is negative at $t = 0$, then increases over time and eventually approaches $\rho = 0.02$ (see panel C).

The Ramsey planner front loads consumption so much that increases in output $N_{0,t}^* = C_{0,t}^* + G_t$ dominate decreases in the tax rate $\tau_{0,t}^* = 1 - C_{0,t}^*$, making the primary surplus $S_{0,t}^* = \tau_{0,t}^* N_{0,t}^* - G_t = (1 - C_{0,t}^*)(C_{0,t}^* + G_t) - G_t$ increase over time. Additionally, as $b_{0,t}$ decreases over time, the wedge $S_{0,t}^* - b_{0,t}$ is significantly negative initially, turns positive in $t = 5.8$, and ultimately converges to 0.7% (see panel D).

By inducing an increasing equilibrium $r_{0,t}^*$ path (panel C), the planner lowers the time-0 value of its wedge $S_{0,t}^* - b_{0,t}$ shown in panel D. As a result, the cumulative liability $\Pi_t^* = \int_0^t e^{\int_0^u r_{0,v}^* dv} (S_{0,u}^* - b_{0,u}) du$ increases (panel E). Finally, the state variable $X_t^* = \Pi_t^* U_{C,t}^*$ also increases over time (panel F). This is because both Π_t^* and $U_{C,t}^*$ increase over time.

7.2 $\{b_{0,t}\}$ is Cyclical

This subsection analyzes an example where the initial debt structure $\{b_{0,t}\}$ is cyclical. Figure 7 plots the solution for the case where $b_{0,t} = b_0 + \theta_b \sin(\frac{\pi}{2}t)$ with $b_0 = 0.1$ and $\theta_b = 0.01$ for $t \geq 0$.

Under the Ramsey plan, the tax rate $\tau_{0,t}^*$ is cyclical over time; it reaches a peak whenever the $\{b_{0,t}\}$ reaches a trough (see panel B). As $C_{0,t}^* = 1 - \tau_{0,t}^*$, the equilibrium consumption process has the same peaks and troughs as the $\{b_{0,t}\}$ process.⁹

The equilibrium interest rate is $r_{0,t}^* = \rho + g_{0,t}^*$ where $g_{0,t}^* = \dot{C}_{0,t}^*/C_{0,t}^*$ is the equilibrium consumption growth rate, which is also cyclical with the same period as but different peaks/troughs from those for $b_{0,t}$ and $\tau_{0,t}^*$. We mark the peaks/troughs for $r_{0,t}^*$ with triangles to indicate that they are different from the peaks/troughs in panels A, B, and D for $b_{0,t}$, $\tau_{0,t}^*$, and the wedge between primary surplus and $b_{0,t}$: $S_{0,t}^* - b_{0,t}$.¹⁰ Panel D also shows that the primary surplus exceeds $b_{0,t}$ when $b_{0,t} < 0.1$ and otherwise when $b_{0,t} > 0.1$.

Making the tax rate and the wedge $S_{0,t}^* - b_{0,t}$ processes move in the opposite direction as the $b_{0,t}$ process is optimal because the marginal utility (and hence bond price) is low when $\{b_{0,t}\}$ is high, thus reducing the value of the government's cumulative liability and achieving the constrained welfare maximization.

⁹In this example, $C_{0,t}^* = \left(\sqrt{1 + 4(b_0 + \theta_b \sin(\frac{\pi}{2}t)) \Phi_0^*(1 + \Phi_0^*)} - 1 \right) / (2(1 + \Phi_0^*))$, which is cyclical with the same period as $b_{0,t}$. The Lagrangian multiplier is $\Phi_0^* = 0.7256$.

¹⁰Note that this wedge shares the same peaks and troughs as the tax rate $\{\tau_{0,t}^*\}$ process.

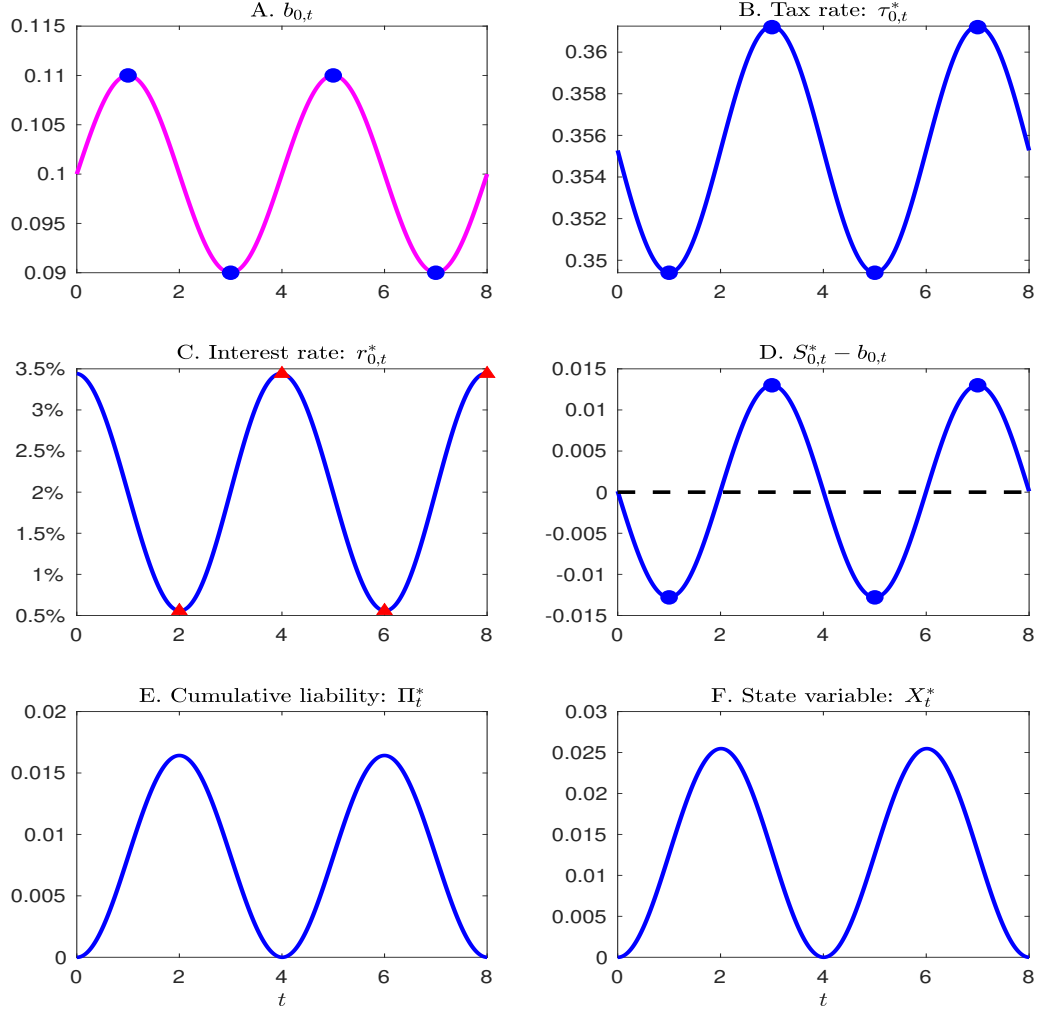


Figure 7: Ramsey plan for cyclical $b_{0,t} = b_0 + \theta_b \sin\left(\frac{\pi}{2}t\right)$. Parameter values: $b_0 = 0.1, \theta_b = 0.01, G_t = 0.2, \rho = 0.02$, and $\eta = 1$.

Finally, panels E and F show that the implied cumulative liability Π_t^* and the state variable $X_t = \Pi_t^* U_{C,t}^*$ are also cyclical but with different peaks/troughs as the underlying $b_{0,t}$ process.

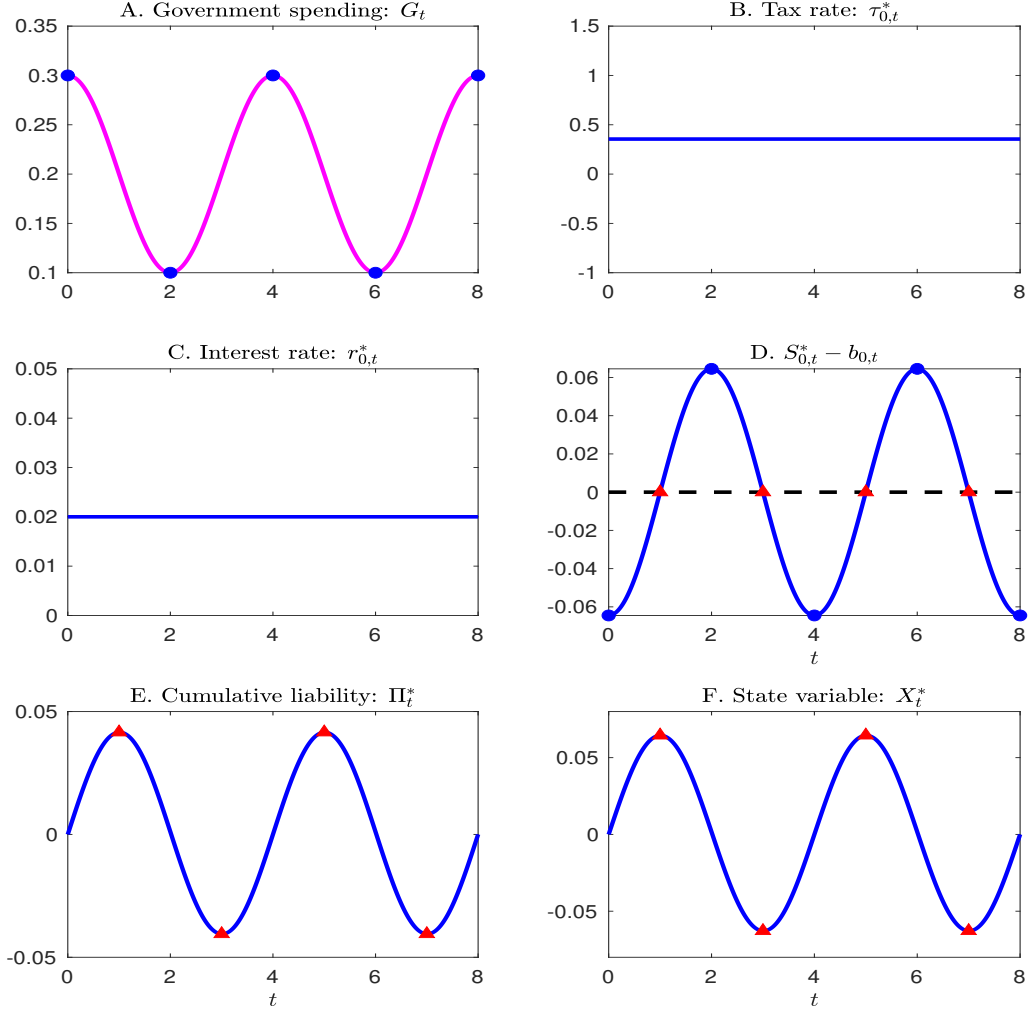


Figure 8: Ramsey plan for cyclical spending $G_t = G_0 + \theta_G \cos\left(\frac{\pi}{2}t\right)$. Parameter values: $G_0 = 0.2, \theta_G = 0.1, b_{0,t} = 0.1, \rho = 0.02$, and $\eta = 1$.

7.3 $\{G_t\}$ is Cyclical

Figure 8 presents an example in which the government finances a cyclical government spending process and a constant $b_{0,t}$ process. The Ramsey planner chooses a constant flat tax rate process. This is because $\tau_{0,t}^* = 1 - C_{0,t}^*$ and $C_{0,t}^* = 0.64$. Outcomes resemble those in Barro (1979), in which the intertemporal properties of the optimal tax rate are disconnected from the intertemporal properties of that exogenous government expenditure process that

is to be financed. Moreover, the equilibrium interest rate process $r_{0,t}^* = \rho + \dot{C}_{0,t}^*/C_{0,t}^* = \rho$ (see panel C) as consumption $C_{0,t}^*$ is constant. Panel D shows that the primary surplus process $S_{0,t}^*$ is also cyclical, reaching a peak (trough) when the G_t process reaches a trough (peak). Solid dots highlight the troughs/peaks in panels A and D. Triangles in panel D highlights points where $S_{0,t}^* = b_{0,t} = 0.1$.

Panel E plots the cumulative liability $\Pi_t^* = \int_0^t e^{\rho u} (S_{0,u}^* - b_{0,u}) du$, which reaches a peak (trough) when $S_{0,t}^* = b_{0,t} = 0.1$.¹¹ Finally, the state variable $X_t^* = \Pi_t^* U_{C,t}^*$ is also cyclical that has the same peaks and troughs as Π_t^* because constant consumption implies constant $U_{C,t}^*$ (see the triangles in panels D, E, and F.)

Our technical machinery also applies when $\{b_{0,t}, G_t\}$ are discontinuous. In the next two sections, we describe implementations of Ramsey plans in settings with Càdlàg $\{b_{0,t}\}$ processes. Section 8 shows how to use instantaneous debt to finance the cumulative liability Π_t^* . Section 9 connects our implementation for Càdlàg $\{b_{0,t}, G_t\}$ processes to appropriate limits of continuous $\{b_{0,t}, G_t\}$ processes. We show how settings with continuous $\{b_{0,t}, G_t\}$ processes in Section 7 can approximate Ramsey plans for discontinuous $\{b_{0,t}, G_t\}$ processes.

8 Financing Deficits

Under a Ramsey plan, the equilibrium bond price $\{q_{0,t}^*; t \geq 0\}$ satisfies $q_{s,t}^* = e^{-\int_s^t r_{0,u}^* du}$ and the associated “force of interest” (the instantaneous interest rate) $r_{0,t}$ is

$$r_{0,t}^* = \begin{cases} -\frac{d \ln q_{0,t}^*}{dt}, & t \notin \mathcal{T}, \\ -\ln \left(\frac{q_{0,t}^*}{q_{0,t-}^*} \right) \delta(t - t_i), & t = t_i \in \mathcal{T}, \end{cases} \quad (48)$$

where $\delta(\cdot)$ is the Dirac function.¹²

General debt management equation.

Let B_t denote the instantaneous debt balance at t . Interest payments over an infinitesimal time interval $(t, t + dt)$ are $r_{0,t}^* B_t dt$. The government uses both instantaneous debt and term

¹¹This follows from $\frac{d\Pi_t^*}{dt} = e^{\rho t} (S_{0,t}^* - b_{0,t})$.

¹²A Dirac function can be considered as the differential of the Heaviside step function (Kanwal, 2012): $h'(t - t_i) \equiv \delta(t - t_i)$, where $h(\cdot)$ is the Heaviside step function: $h(t) = 0, \forall t < 0$ and $h(t) = 1, \forall t \geq 0$.

debt to finance its primary deficits, so the following dynamic budget constraint holds:¹³

$$d\Pi_t^* = dB_t + \underbrace{\left(\int_{s \geq t} q_{t,s}^* \partial_t b_{t,s} ds \right) dt}_{\text{term debt issued at } t} - \underbrace{\left(\int_0^t \partial_u b_{u,t} du \right) dt}_{\text{term debt due at } t}, \quad (49)$$

where Π_t^* defined in (15) is the cumulative liabilities under the Ramsey plan absent term debt adjustments. Here $\partial_u b_{u,t}$ denotes a rate of increase the (incremental) debt coupon issued at time u and due at t . The third term $\left(\int_0^t \partial_u b_{u,t} du \right) dt$ is the (incremental) term debt due at t , issued cumulatively from 0 to t . The second term is the net financing from new debt issuance over dt . the change of instantaneous debt balance dB_t finances shortfalls not financed by term debt changes.

Managing Π_t^* with instantaneous debt only.

The structure of the contractible subspace allows us to restrict debt restructuring policies to a set in which $\partial_t b_{t,s} = 0$ so that $b_{t,s} = b_{0,s}$ for all $s \geq t, t \geq 0$, so that no term debt is issued at any $t \geq 0$. That in turn means that the instantaneous debt balance B_t equals Π_t^* for all $t \geq 0$. Thus, it is sufficient to only use instantaneous debt to finance the cumulative liabilities that are governed by $d\Pi_t^* = r_{0,t}^* \Pi_t^* dt + (b_{0,t} + G_t - \tau_{0,t}^* N_{0,t}^*) dt$ as given in (32). With $B_0 = \Pi_0^*$, we obtain the following instantaneous debt balance dynamics:

$$dB_t = r_{0,t}^* B_t dt + (b_{0,t} + G_t - \tau_{0,t}^* N_{0,t}^*) dt. \quad (50)$$

Remark 5. *The debt rescheduling policies of Lucas and Stokey (1983) use only term debt. Those policies use a coupon flow management policy of $\{\partial_t b_{t,s}; s \geq t, t \geq 0\}$ and $\{B_t = 0; t \geq 0\}$ to satisfy the dynamic budget constraint:*

$$d\Pi_t^* = \left(\int_{s \geq t} q_{t,s}^* \partial_t b_{t,s} ds \right) dt - \left(\int_0^t \partial_u b_{u,t} du \right) dt. \quad (51)$$

This policy requires continuous adjustments to the term debt adjustment and makes it challenging to formulate the Ramsey problem recursively.

¹³When $t \in \mathcal{T}$, all instantaneous debt and term debt must be re-evaluated under the Ramsey plan from time $t-$ to time t in order to preserve the purchasing power of Π_t^* under the Ramsey plan. That is, the instantaneous debt balance shall change to $B_t = B_{t-} q_{0,t-}^* / q_{0,t}^*$ from B_{t-} and the term debt service flows change to $b_{t,s} = b_{t-,s} q_{0,t-}^* / q_{0,t}^*$ from $b_{t-,s}$ for all $s \geq t$.

When the Ramsey planner has access to instantaneous debt, the Ramsey planner gains nothing by rescheduling term debt because instantaneous debt balance B_t perfectly tracks the cumulative deficits Π_t^* and hence summarizes both price and quantity information about the Ramsey plan. Furthermore, because instantaneous debt has the shortest maturity, it is least subject to future governments' manipulation. In the next section, we describe how a (local) commitment condition that is sufficient to convince continuation Ramsey planners to confirm a Ramsey plan.

9 Implementing a Ramsey Plan

In subsection 9.1, we implement the Ramsey plan with instantaneous debt under a (local) commitment condition that we add to the commitments that Lucas and Stokey (1983) and Debortoli, Nunes, and Yared (2021) impose on continuation planners. In subsection 9.2 we assess the strength of our (local) commitment condition.

9.1 Commitment Condition and Implementation of Ramsey Plan

In our implementation, the government at time $t-$ leaves a debt structure $\{\widehat{B}_t, \widehat{b}_{0,s}; s \geq t\}$ to the time t government, where

$$\widehat{B}_t = \Pi_t^* \quad \text{and} \quad \widehat{b}_{t-,s} = b_{0,s}; \quad s \geq t. \quad (52)$$

Assumption 3 (local commitment condition). *To preserve the purchasing power of the instantaneous debt inherited by a time t government, we impose:*

$$\widehat{B}_t U_{C,t}(C_{t,t}, N_{t,t}) = B_{t-} U_{C,t-}(C_{t-,t-}, N_{t-,t-}), \quad (53)$$

where $C_{t,t}$ and $N_{t,t}$ are the time t government's choices.

Note that if the Ramsey plan has been implemented until time $t-$, then $C_{s,s} = C_{0,s}^*$ and $N_{s,s} = N_{0,s}^*$ for $0 < s < t$. We now state:

Proposition 6. *The Ramsey plan can be implemented by leaving $\{b_{0,t}\}$ untouched and adjusting instantaneous debt according to (52) under the local commitment condition stated in Assumption 3.*

Proof. We use a recursion. Under the assumption the Ramsey plan is implemented up to time $t-$, it is sufficient if we prove that the Ramsey plan is also implemented at time t .¹⁴ We can show that at time t the continuation Ramsey planner, confronting $\widehat{B}_t = \Pi_t^*$ and $\widehat{b}_{t-,s} = b_{0,s}$ for all $s \geq t$, will confirm the continuation of the Ramsey plan by choosing $C_{t,s} = C_{0,s}^*$ and $N_{t,s} = N_{0,s}^*$ for all $s \geq t$. We can prove the preceding claim by subdividing time- t continuation Ramsey planner's problem into two subproblems.

First, the time- t continuation Ramsey planner chooses $\{C_{t,s}, s \geq t\}$ to attain

$$\max_{\{C_{t,s}; s \geq t\}} \int_t^\infty e^{-\rho(s-t)} U(C_{t,s}, C_{t,s} + G_s) ds, \quad (54)$$

where maximization subject to $\widehat{X}_t = \widehat{B}_t U_{C,t}(C_{t,t}, N_{t,t})$ and the following dynamics for $s \geq t$:

$$d\widehat{X}_s = (\rho\widehat{X}_s - C_{t,s}U_{C,s} - N_{t,s}U_{N,s} + b_{0,s}U_{C,s}) ds. \quad (55)$$

Similar to our analysis in Section 6, we can show that the value function for (54), denoted as $\widehat{V}(\widehat{X}_t, t; \Phi_t)$, is additively separable: $\widehat{V}(\widehat{X}_t, t; \Phi_t) = \Phi_t \widehat{X}_t + \widehat{H}(t; \Phi_t)$, where $\widehat{H}(t; \Phi_t)$ solves a differential equation (A-10) in Appendix A, and that $C_{t,s} = C(\Phi_t; b_{0,s}, G_s)$ where $C(\Phi_t; b_{0,s}, G_s)$ is the same as the one given in (41). This is because the initial debt term structure $\{b_{0,t}; t \geq 0\}$ is unchanged in our implementation with instantaneous debt only.

Under the local commitment condition given in Assumption 3, the time t continuation Ramsey planner optimally chooses $C_{t,t} = C_{0,t}^*$ to confirm the continuation of the Ramsey plan because 1.) $\widehat{B}_t = \Pi_t^*$ and $B_{t-} = \Pi_{t-}^*$ under our implementation; 2.) $\Pi_{t-}^* U_{C,t-}(C_{0,t-}^*, N_{0,t-}^*) = \Pi_t^* U_{C,t}(C_{0,t}^*, N_{0,t}^*)$ under the Ramsey plan; and 3.) the local commitment condition: $\widehat{B}_t U_{C,t}(C_{t,t}, N_{t,t}) = B_{t-} U_{C,t-}(C_{0,t-}^*, N_{0,t-}^*)$. So we have proved $\widehat{X}_t = X_t^* = \Pi_t^* U_{C,t}(C_{0,t}^*, N_{0,t}^*)$ and $C_{t,t} = C_{0,t}^*$.

Next, we prove that the time- t continuation Ramsey planner will choose $C_{t,s} = C_{0,s}^*$ for all $s > t$. Because $\{C_{t,s}, b_{0,s}, G_s; s \geq t\}$ are Càdlàg processes and $C_{t,s} = C(\Phi_t; b_{0,s}, G_s)$, we have $\lim_{s \rightarrow t} C_{t,s} = \lim_{s \rightarrow t} C(\Phi_t; b_{0,s}, G_s) = C_{0,t}^* = C(\Phi_0^*; b_{0,t}, G_t)$. Under regularity conditions, we obtain $\Phi_t = \Phi_0^*$ and $C_{t,s} = C(\Phi_t^*; b_{0,s}, G_s) = C_{0,s}^*$ for all $s \geq t$.

Second, the time- t continuation Ramsey planner has to set Φ_t to Φ_0^* (for the Ramsey plan) given in Proposition 4 to satisfy the time t implementability condition:

$$\int_t^\infty e^{-\rho(s-t)} (C_{t,s}U_{C,s} + (C_{t,s} + G_s)U_{N,s}) ds = \int_t^\infty e^{-\rho(s-t)} b_{0,s}U_{C,s} dt + \widehat{B}_t U_{C,t} \quad (56)$$

¹⁴At $t = 0$, the Ramsey planner is also a time 0 continuation Ramsey planner.

when maximizing $\widehat{V}(\widehat{X}_t, t; \Phi_t)$. This follows because (56) holds when $\Phi_t = \Phi_0^*$. \square

9.2 Strength of Local Commitment Condition

To understand the strength of our local commitment condition, we begin by studying special settings in which the initial debt term structure and government expenditure processes $\{b_{0,t}, G_t; t \geq 0\}$ are continuous in time. In these settings, the local-commitment condition boils down to requiring that the consumption is continuous in t . The class of $\{b_{0,t}, G_t; t \geq 0\}$ processes that are continuous in time unleash the key economic forces that also shape Ramsey plans in more general settings in which $\{b_{0,t}, G_t; t \geq 0\}$ are Càdlàg processes, such as the continuous-time formulation of Debortoli, Nunes, and Yared (2021)'s counterexample presented in section 4. We verify this claim by noting how a Ramsey plan in a generic Càdlàg setting can be approximated arbitrarily well by a Ramsey plan in a continuous $\{b_{0,t}, G_t; t \geq 0\}$ process setting.

9.2.1 Continuous $\{b_{0,t}, G_t; t \geq 0\}$ Settings

Using Lemma 4, we show that in settings satisfying Assumption 2, the local commitment condition can be expressed in another way.

Lemma 6. *Under Assumption 2, the local commitment condition (53) under our instantaneous-debt-based implementation is equivalent to requiring that consumption is continuous in time t :*

$$C_{t,t} = C_{t-,t-}. \quad (57)$$

Proof. We use a recursion. When (57) holds up to time $t-$, it is enough to show that our local commitment condition is satisfied at time t . Under our implementation, choosing $\widehat{B}_t = \Pi_t^*$ at time t implies that the instantaneous debt balance is continuous at t :

$$\widehat{B}_t = \Pi_t^* = \Pi_{t-}^* = B_{t-}, \quad (58)$$

where the second equality follows from the result that Π_t^* is continuous in t under Ramsey plan (Lemma 4) and the third equality uses the result that the instantaneous debt balance is given by $B_{t-} = \Pi_{t-}^*$ under our implementation. Substituting (58) into the local commitment condition (53), we obtain (57). \square

Thus, when $b_{0,t}$ and G_t are continuous in time t , our local commitment condition amounts to requiring that consumption is continuous in time t . Recall that in the Ramsey

plan $\tau_{0,t}^* = 1 + U_N(C_{0,t}^*, N_{0,t}^*)/U_C(C_{0,t}^*, N_{0,t}^*)$. Since the tax rate is continuous in t , using instantaneous debt implements the Ramsey plan.¹⁵

9.2.2 Càdlàg $\{b_{0,t}, G_t; t \geq 0\}$ Processes

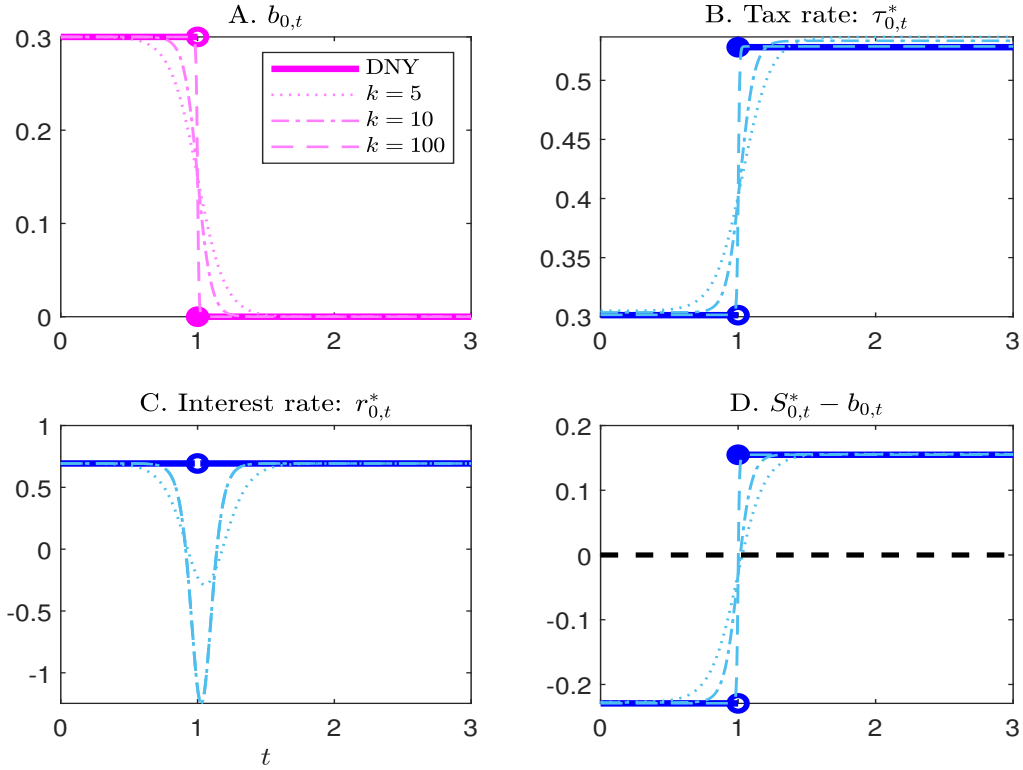


Figure 9: Approximating Ramsey plan in Debortoli et al.’s $b_0 = 0.3$ example. A process $b_{0,t}^{(k)} = b_0(1 + e^{2k(t-1)})^{-1}$ with $k = 100$ does a good job of approximating the $\{b_{0,t}\}$ process in equation (17). Parameter values are $G_t = 0.2$, $e^{-\rho} = 0.5$, and $\eta = 1$.

In this section we study Càdlàg settings in which $\{b_{0,t}\}$ and/or $\{G_t\}$ processes are discontinuous in time t . In these settings, the local commitment condition requires instantaneous debt balance to jump at discontinuity points, so that neither (57) nor (58) holds. This outcome is a consequence of the Ramsey plan’s requiring $U_{C,t}$ to jump at discontinuities of $\{b_{0,t}\}$ and/or $\{G_t\}$. At those discontinuity points, the interest rate $r_{0,t}$ is a Dirac delta

¹⁵In continuous $\{b_{0,t}, G_t; t \geq 0\}$ settings, our local commitment condition requires the consumption rate to be continuous. Thus, it allows consumption to vary over every finite time interval. One way to interpret discrete-time models is that consumption flows in an underlying continuous-time models are restricted to be constant within each time period.

function, an equilibrium outcome of our closed-economy setting that eliminates arbitrage opportunities. This outcome is associated with our expansion of the contractible subspace relative to that in Lucas and Stokey (1983). It is part and parcel of complete markets.

For most of this paper we chose to stay as close as possible to discrete-time settings like Debortoli, Nunes, and Yared (2021)'s by focusing on settings with Càdlàg in which the initial debt service process $\{b_{0,t}\}$ has discontinuities. In such settings, the interest rate process is a Dirac delta function that “equals” ∞ at discontinuity points. But a discontinuous $\{b_{0,t}\}$ process can be well approximated by the limit of a sequence of continuous $\{b_{0,t}\}$ processes (see Proposition 7), so studying settings with continuous $\{b_{0,t}\}$ and $\{G_t\}$ processes lets us isolate forces that shape a Ramsey plan. Figures 9 and 10 illustrate for both Debortoli et al.'s $b_0 = 0.3$ and $b_0 = 0.5$ examples analyzed in Section 4.¹⁶

Proposition 7. *Let $\{b_{0,t}^{(k)}, G_t^{(k)}; t \geq 0\}$, $k = 1, 2, \dots$, be a sequence of continuous processes that converges uniformly to Càdlàg processes $\{b_{0,t}, G_t; t \geq 0\}$. Let $\{C_{0,t}^{*,(k)}, N_{0,t}^{*,(k)}; t \geq 0\}$ and $\{C_{0,t}^*, N_{0,t}^*; t \geq 0\}$ be corresponding Ramsey plans. Under Appendix A regularity conditions:*

$$\lim_{k \rightarrow \infty} C_{0,t}^{*,(k)} \rightarrow C_{0,t}^*, \quad \lim_{k \rightarrow \infty} N_{0,t}^{*,(k)} \rightarrow N_{0,t}^* \quad \text{for } t \geq 0. \quad (59)$$

10 Concluding Remarks

Because our government has access to instantaneous debt while Lucas and Stokey's and Debortoli et al.'s governments don't, our implementation of a Ramsey plan can work even when theirs won't. In Debortoli et al.'s counterexample, a budget feasible restructuring of government debt from $\{b_{0,t}\}$ to $\{\hat{b}_{0,t}\}$ can't motivate future governments to implement the Ramsey tax plan. We have demonstrated how appropriate management of instantaneous debt together with a local (instantaneous) commitment does motivate them to implement the Ramsey tax plan.

Along with Aguiar et al. (2019), our Ramsey planner assigns short-term government debt a special role. But ours is a closed economy in which the planner manipulates equilibrium interest rates. Disparate economic forces make short-term debt central in Aguiar et al.'s model and ours. In our model, the Ramsey planner is a Stackelberg leader who

¹⁶Our analysis of Ramsey plans in settings with continuous $\{b_{0,t}\}$ and $\{G_t\}$ processes can help us understand Ramsey plans and their implementations from discrete-time models. Thus, start with a Ramsey plan in a setting with continuous $\{b_{0,t}\}$ and $\{G_t\}$ processes. For a finite time interval Δ , construct an associated discrete-time model by forming a discrete time consumption process $\{C_{0,t}\Delta\}_{t=0}^{\infty}$ and so on. The discrete time consumption process is “discontinuous in t ” even though the consumption rate $C_{0,t}$ is continuous.

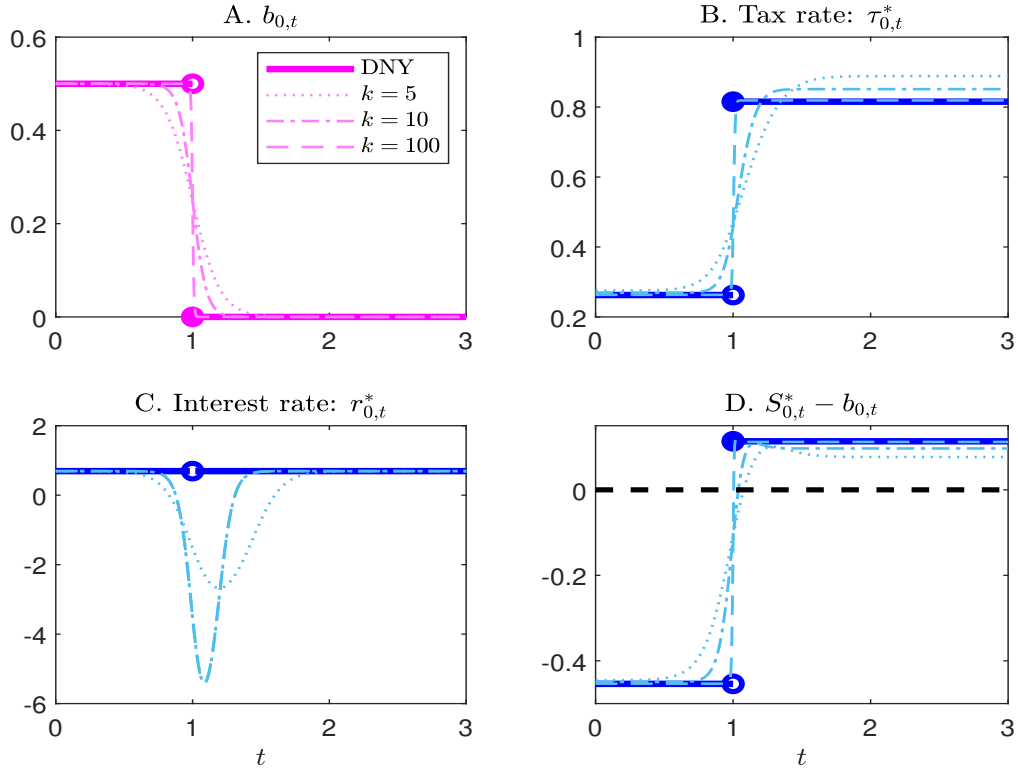


Figure 10: Approximating Ramsey plan in Debortoli et al.’s $b_0 = 0.5$ example. This figures shows that using $b_{0,t}^{(k)} = b_0(1 + e^{2k(t-1)})^{-1}$ with $k = 100$ well approximates $b_{0,t}$ given in equation (17). Parameter values are $G_t = 0.2, e^{-\rho} = 0.5$ and $\eta = 1$.

manipulates bond prices to its advantage. Instantaneous debt is a powerful tool for the Ramsey planner to constrain continuation Ramsey planners in ways that facilitate implementing the Ramsey plan. Because the maturity of instantaneous debt is infinitesimal, its value is least vulnerable to manipulation by future governments. A local commitment condition suffices to prevent continuation government from diluting the purchasing power of instantaneous debt, inducing them to confirm a plan chosen by the Ramsey planner at time 0.¹⁷

Our analysis of a continuous time version of Debortoli et al.’s model sets the stage for adding shocks and complete markets in state-contingent securities in the spirit of Lucas and Stokey. In future work, we hope to extend our Sections 8 - 9 analysis to study im-

¹⁷Debortoli et al.’s counterexample reveals that Lucas and Stokey’s way of restructuring the term structure of government debt is sometimes unable to deter future governments from manipulating bond prices by deviating from a Ramsey tax plan. This finding is related to incentives to overproduce in the durable goods monopoly problem studied by Coase (1972) and Stokey (1981).

plementable Ramsey plans in settings with random $\{G_t\}$ processes. That should help us connect the presence of instantaneous debt and our local commitment condition to assumptions about contractible spaces adopted in the complete markets formulations of Black and Scholes (1973) and Harrison and Kreps (1979).

We close by noting that the US Treasury and the Congressional Budget Office have gone part-way toward the arrangement that we have used to implement the Ramsey plan. Although today it doesn't issue a counterpart to instantaneous debt B_t as the government in our model does, it books and reports accumulated past primary deficits Π_t^* .

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Appendices

A Technical Details

Proof of Lemma 2. We adapt Debortoli et al.'s proof to our continuous-time setting. We focus on settings in which the utility function is given in (16) for $\eta > 0, \gamma = 1$, the initial debt structure of (17), and constant government spending as in Debortoli et al. (2021).

We first derive the unique debt restructuring policy that may induce the continuation Ramsey planner to confirm the Ramsey plan. Following the procedure outlined in Section 4, we obtain debt restructuring policy (24).

Recall that

$$C(\Lambda_0; b_0, G) = \frac{2b_0\Lambda_0}{\sqrt{1 + 4\eta b_0\Lambda_0(1 + \Lambda_0)} - 1}. \quad (\text{A-1})$$

We can show that under the Ramsey plan C_1^* given in (21) is a decreasing function of b_0 and \hat{b}_1/C_1^* is an increasing function of b_0 . This is because 1) the function $C(\Lambda_0; 0, G)$ obtained by substituting $b_0 = 0$ into (A-1) is decreasing in Λ_0 ; 2) $\eta(C(\Lambda_0; b_0, G) + G) + \frac{b_0}{C(\Lambda_0; b_0, G)}$ is an increasing function of b_0 when $b_0 \geq 0$; and 3) Λ_0^* satisfies:

$$\frac{1 - e^{-\rho T}}{\rho} \left(1 - \eta(C(\Lambda_0^*; b_0, G) + G) - \frac{b_0}{C(\Lambda_0^*; b_0, G)} \right) + \frac{e^{-\rho T}}{\rho} (1 - \eta(C(\Lambda_0^*; 0, G) + G)) = 0. \quad (\text{A-2})$$

Applying the implicit function theorem to (A-2), implies that $d\Lambda_0^*/db_0 < 0$ because for a fixed Λ_0 , the first part of (A-2) decreases in b_0 and $C(\Lambda_0; 0, G)$ decreases in Λ_0 . Equation (A-2) requires that Λ_0^* increases with b_0 . Consequently, $C_1^* = C(\Lambda_0^*; 0, G)$ decreases in b_0 and $\frac{\hat{b}_1}{C_1^*} = (e^{\rho T} - 1) \left(\frac{b_0}{C_0^*} - 1 + \eta(C_0^* + G) \right)$ increases in b_0 .

Next, we formulate the problem facing a continuation Ramsey planner at time $t = T$. Let Λ_T denote a Lagrange multiplier. For the continuation Ramsey planner to want to confirm the Ramsey plan, the Lagrange multiplier Λ_T for the continuation Ramsey plan must satisfy:

$$\Lambda_T = -\frac{1 - \eta C_1^*}{\hat{b}_1/C_1^* - \eta C_1^*} = -\frac{1 - \eta C_1^*}{1 - \eta(C_1^* + G) - \eta C_1^*}. \quad (\text{A-3})$$

The first equality in (A-3) is implied by the first order condition for the Ramsey plan and the second equality in (A-3) follows from $\hat{b}_1/C_1^* = 1 - \eta(C_1^* + G)$, which is implied by the

budget constraint under the Ramsey plan.

Finally, note that when b_0 is too high, the implied C_1^* is too low, and Λ_T that satisfies (A-3) can be negative. That make continuing the Ramsey plan suboptimal for the continuation Ramsey planner. Therefore, there exists a b^* such that only when $b_0 \leq b^*$ so that the Lagrangian multiplier Λ_T is positive. □

Regularity Conditions for Proposition 2. We provide two regularity conditions that guarantee that a Ramsey plan exists. First, for a given initial debt structure, the admissible allocation set is not empty, so there exists allocations $\{C_{0,t}; t \geq 0\}$ that satisfy the implementability condition (7). Second, $(CU_C + (C + G)U_N - bU_C)$ is concave for $G > 0, b \geq 0$.

For economies analyzed in Debortoli, Nunes, and Yared (2021) and this paper (with a utility function given in (16) and an initial debt structure given by (17)), these two regularity conditions are satisfied when b_0 is not too high. □

Proof of Proposition 3. Given Φ_0 , consumption at t , $C_{0,t} = C(\Phi_0; b_{0,t}, G_t)$, solves

$$\max_{C_{0,t}} U(C_{0,t}, C_{0,t} + G_t) + \Phi_0 (C_{0,t}U_{C,t} + (C_{0,t} + G_t)U_{N,t} - b_{0,t}U_{C,t}). \quad (\text{A-4})$$

First we can verify that under the preceding regularity conditions a solution $C_{0,t} > 0$ exists for (A-4). Second, we can show that $C_{0,t} = C(\Phi_0; b_{0,t}, G_t)$ satisfies the following first-order necessary condition for problem (A-4):

$$(1 + \Phi_0)(U_{C,t} + U_{N,t}) + \Phi_0 [(C_{0,t} - b_{0,t})(U_{CC,t} + U_{CN,t}) + (C_{0,t} + G_t)(U_{NC,t} + U_{NN,t})] = 0, \quad (\text{A-5})$$

The associated labor supply is $N(\Phi_0; b_{0,t}, G_t) = C(\Phi_0; b_{0,t}, G_t) + G_t$.

Recall $V(X_t, t) = -\Phi_0 X_t + H(t; \Phi_0)$ (see (39) in subsection 6.2.) Now solve ODE (40) for $H(t; \Phi_0)$. Using $C(\Phi_0; b_{0,t}, G_t)$ and $N(\Phi_0; b_{0,t}, G_t)$, we can rewrite (40) as

$$d(-e^{-\rho t} H(t; \Phi_0)) = e^{-\rho t} \left[U(C(\Phi_0; b_{0,t}, G_t), N(\Phi_0; b_{0,t}, G_t)) + \Phi_0 (N(\Phi_0; b_{0,t}, G_t)U_{C,t} + N(\Phi_0; b_{0,t}, G_t)U_{N,t} - b_{0,t}U_{C,t}) \right] dt.$$

Integrating the above equation from t to ∞ and using the transversality condition

$\lim_{t \rightarrow \infty} e^{-\rho t} H(t; \Phi_0) = 0$, we obtain

$$H(t; \Phi_0) = \int_t^\infty e^{-\rho(s-t)} \left[U(C(\Phi_0; b_{0,s}, G_s), N(\Phi_0; b_{0,s}, G_s)) \right. \\ \left. + \Phi_0 (C(\Phi_0; b_{0,s}, G_s)U_{C,s} + N(\Phi_0; b_{0,s}, G_s)U_{N,s} - b_{0,s}U_{C,s}) \right] dt.$$

Therefore, the value function $V(X_t, t)$ defined for the DP problem in Definition 3 satisfies

$$\begin{aligned} V(X_t, t) &= -\Phi_0 X_t + H(t; \Phi_0) \\ &= -\Phi_0 \Pi_t U_{C,t} + \int_t^\infty e^{-\rho(s-t)} U(C(\Phi_0; b_{0,s}, G_s), N(\Phi_0; b_{0,s}, G_s)) ds \end{aligned} \quad (\text{A-6})$$

$$\begin{aligned} &+ \Phi_0 \int_t^\infty e^{-\rho(s-t)} (C(\Phi_0; b_{0,t}, G_t)U_{C,t} + N(\Phi_0; b_{0,t}, G_t)U_{N,t} - b_{0,t}U_{C,t}) ds \\ &= \int_t^\infty e^{-\rho(s-t)} U(C(\Phi_0; b_{0,s}, G_s), N(\Phi_0; b_{0,s}, G_s)) ds. \end{aligned} \quad (\text{A-7})$$

Equation (A-7) follows from equation (A-6) because the time-0 budget constraint and the definition of Π_t in (31) together imply that the sum of the first and third terms in (A-6) equals zero. \square

Technical Details for Proposition 6. Here we describe some details involved in solving the time t continuation Ramsey planner's dynamic problem. The value function $\widehat{V}(\widehat{X}_s, s; \Phi_t)$ at s for $s \geq t$ satisfies the Bellman equation:

$$\rho \widehat{V} = \frac{\partial \widehat{V}}{\partial s} + \max_{C_{t,s}} U(C_{t,s}, C_{t,s} + G_s) + \frac{\partial \widehat{V}}{\partial X_s} (\rho X_s - C_{t,s}U_{C,s} - (C_{t,s} + G_t)U_{N,s} + b_{0,s}U_{C,s}). \quad (\text{A-8})$$

Note that it is Φ_t not Φ_s that appears in $\widehat{V}(\widehat{X}_s, s; \Phi_t)$. We guess and verify that $\widehat{V}(\widehat{X}_s, s; \Phi_t)$ is additively separable in \widehat{X}_s and s :

$$\widehat{V}(\widehat{X}_s, s; \Phi_t) = -\Phi_t \widehat{X}_s + \widehat{H}(s; \Phi_t); \quad s \geq t. \quad (\text{A-9})$$

Substituting (A-9) into (A-8), we obtain the following for $\widehat{H}(s; \Phi_t)$ where $s \geq t$:

$$\rho \widehat{H}(s; \Phi_t) = \widehat{H}'(s; \Phi_t) + \max_{C_{t,s}} U(C_{t,s}, C_{t,s} + G_s) + \Phi_t (C_{t,s}U_{C,s} + (C_{t,s} + G_s)U_{N,s} - b_{0,s}U_{C,s}). \quad (\text{A-10})$$

Integrating (A-10) from t to ∞ and using the transversality condition, we obtain:

$$\begin{aligned} \hat{H}(s; \Phi_t) &= \int_s^\infty e^{-\rho(u-s)} [U(C(\Phi_t; b_{0,u}, G_u), C(\Phi_t; b_{0,u}, G_u) + G_u) \\ &\quad + \Phi_t (C(\Phi_t; b_{0,u}, G_u)U_{C,u} + (C(\Phi_t; b_{0,u}, G_u) + G_u)U_{N,u} - b_{0,u}U_{C,u})] du. \end{aligned} \quad (\text{A-11})$$

□

Proof of Proposition 7. First, we impose two regularity conditions: 1) The utility function $U(\cdot, \cdot)$ and the associated function $C(\cdot; \cdot, \cdot)$ given in (41) are continuous in all arguments; 2) as functions of time t , $e^{-\rho t}U(C(\Phi_0; b_{0,t}^{(k_i)}, G_t^{(k_i)}), C(\Phi_0; b_{0,t}^{(k_i)}, G_t^{(k_i)} + G_t^{(k_i)})$ and $e^{-\rho t} \left(C(\Phi_0; b_{0,t}^{(k_i)}, G_t^{(k_i)})U_{C,t} + (C(\Phi_0; b_{0,t}^{(k_i)}, G_t^{(k_i)} + G_t^{(k_i)})U_{N,t} - b_{0,t}^{(k_i)}U_{C,t} \right)$ for $k = 1, 2, \dots$ are bounded by an integrable function of time t . The subsection 9.2.2 analysis of Debortoli et al.'s examples satisfy these mild regularity conditions (see Figures 9 and 10.)

Recall that Proposition 4 characterizes the Ramsey plan, when it exists. Thus, for $k = 1, 2, \dots, \infty$, there exist $\Phi_0^{*,(k)} \geq 0$ and $\Phi_0^* \geq 0$ such that $C_{0,t}^{*,(k)} = C(\Phi_0^{*,(k)}; b_{0,t}^{(k)}, G_t^{(k)})$ and $C_{0,t}^* = C(\Phi_0^*; b_{0,t}, G_t)$. Via contradiction we can prove that $\lim_{k \rightarrow \infty} \Phi_0^{*,(k)} = \Phi_0^*$, which implies approximation (59).

Suppose that for a sequence $\{k_1, \dots, k_i, \dots\}$ a non-negative limit of $\Phi_0^{*,(k_i)}$ exists that does not equal Φ_0^* : $\lim_{i \rightarrow \infty} \Phi_0^{*,(k_i)} = \Phi_0' \neq \Phi_0^*$. We can then construct a sequence of consumption allocations $\{C(\Phi_0^*; b_{0,t}^{(k_i)}, G_t^{(k_i)}); t \geq 0\}$ for $i = 1, 2, \dots$. Note first that $\{C(\Phi_0^*; b_{0,t}^{(k_i)}, G_t^{(k_i)})\}$ is an admissible consumption policy. We can show this by integrating dX_t in (35) and then using the transversality condition to verify that $\{C(\Phi_0^*; b_{0,t}^{(k_i)}, G_t^{(k_i)})\}$ satisfies the implementability condition (7).

Second, in a Ramsey plan a $\{C(\Phi_0^*; b_{0,t}^{(k_i)}, G_t^{(k_i)})\}$ policy must yield a smaller households' utility than the Ramsey plan $\{C(\Phi_0^{*,(k_i)}; b_{0,t}^{(k_i)}, G_t^{(k_i)})\}$:

$$\begin{aligned} &\int_0^\infty e^{-\rho t} U(C(\Phi_0^{*,(k_i)}; b_{0,t}^{(k_i)}, G_t^{(k_i)}), C(\Phi_0^{*,(k_i)}; b_{0,t}^{(k_i)}, G_t^{(k_i)} + G_t) dt \\ &> \int_0^\infty e^{-\rho t} U(C(\Phi_0^*; b_{0,t}^{(k_i)}, G_t^{(k_i)}), C(\Phi_0^*; b_{0,t}^{(k_i)}, G_t^{(k_i)} + G_t) dt. \end{aligned} \quad (\text{A-12})$$

Under our regularity conditions, using $\lim_{i \rightarrow \infty} C(\Phi_0^*; b_{0,t}^{(k_i)}, G_t^{(k_i)}) = C(\Phi_0^*; b_{0,t}, G_t)$ and applying the dominated convergence theorem to (A-12), we obtain:

$$\int_0^\infty e^{-\rho t} U(C(\Phi_0'; b_{0,t}, G_t), C(\Phi_0'; b_{0,t}, G_t) + G_t) dt$$

$$\geq \int_0^\infty e^{-\rho t} U(C(\Phi_0^*; b_{0,t}, G_t), C(\Phi_0^*; b_{0,t}, G_t) + G_t) dt. \quad (\text{A-13})$$

The implementability condition under the Ramsey plan is

$$\int_0^\infty e^{-\rho t} \left(C(\Phi_0^{*,(k_i)}; b_{0,t}^{(k_i)}, G_t^{(k_i)}) U_{C,t} + (C(\Phi_0^{*,(k_i)}; b_{0,t}^{(k_i)}, G_t^{(k_i)}) + G_t) U_{N,t} - b_{0,t}^{(k_i)} U_{C,t} \right) dt = 0. \quad (\text{A-14})$$

Under the regularity conditions, applying the dominated convergence theorem to (A-14), we obtain:

$$\int_0^\infty e^{-\rho t} \left(C(\Phi_0'; b_{0,t}, G_t) U_{C,t} + (C(\Phi_0'; b_{0,t}, G_t) + G_t) U_{N,t} - b_{0,t} U_{C,t} \right) dt = 0, \quad (\text{A-15})$$

which follows from $\lim_{i \rightarrow \infty} \Phi_0^{*,(k_i)} = \Phi_0' \neq \Phi_0^*$, $\lim_{i \rightarrow \infty} b_{0,t}^{(k_i)} = b_{0,t}$, and $\lim_{i \rightarrow \infty} G_t^{(k_i)} = G_t$. Therefore, $\{C(\Phi_0'; b_{0,t}, G_t); t \geq 0\}$ is an admissible policy for the initial debt structure $\{b_{0,t}; t \geq 0\}$ and government spending process $\{G_t; t \geq 0\}$. However, the optimality of Φ_0^* require that

$$\begin{aligned} & \int_0^\infty e^{-\rho t} U(C(\Phi_0^*; b_{0,t}, G_t), C(\Phi_0^*; b_{0,t}, G_t) + G_t) dt \\ & > \int_0^\infty e^{-\rho t} U(C(\Phi_0'; b_{0,t}, G_t), C(\Phi_0'; b_{0,t}, G_t) + G_t) dt, \end{aligned} \quad (\text{A-16})$$

which contradicts (A-13). □

B Cyclical Ramsey Plans

B.1 Cyclical $b_{0,t}$ in Subsection 7.2

When $b_{0,t} = b_0 + \theta_b \sin\left(\frac{\pi}{2}t\right)$, G_t is constant, and a utility function given in (16) with $\gamma = 1, \eta = 1$, $C_{0,t}^*$ under the Ramsey plan given by (41) can be simplified to

$$C_{0,t}^* = \left(\sqrt{1 + 4(b_0 + \theta_b \sin\left(\frac{\pi}{2}t\right) \Phi_0^*(1 + \Phi_0^*) - 1)} \right) / (2(1 + \Phi_0^*)), \quad (\text{A-17})$$

which is periodic with the same peaks, troughs, and period (four) as the initial debt structure $b_{0,t}$. The tax rate $\tau_{0,t}^* = 1 - C_{0,t}^*$ also has period (four) but is countercyclical to $b_{0,t}$. When G_t is constant over time, the primary surplus $S(C_{0,t}^*) = \tau_{0,t}^* N_{0,t}^* - G_t = C_{0,t}^* (1 - C_{0,t}^* - G_t)$ is a periodic function with same periodicity as $\tau_{0,t}^*$. This follows from

$$\frac{d(S_{0,t}^* - b_{0,t})}{dt} = (1 - 2C_{0,t}^* - G) \frac{dC_{0,t}^*}{dt} - \frac{db_{0,t}}{dt}$$

and $\frac{dC_{0,t}^*}{dt} = 0$ if and only if $\frac{db_{0,t}}{dt} = 0$.

Table 2: Peaks and troughs in Figure 7.

	$t = 1$	$t = 3$
Initial debt $b_{0,t}$	Peak	Trough
Consumption $C_{0,t}^*$	Peak	Trough
Tax rate $\tau_{0,t}^*$	Trough	Peak
$S(C_{0,t}^*) - b_{0,t}$	Trough	Peak

B.2 Cyclical G_t in Subsection 7.3

When $G_t = G_0 + \theta_G \cos(\frac{\pi}{2}t)$, $b_{0,t} = b_0$ is constant, and utility is (16) with $\gamma = 1, \eta = 1$, $C_{0,t}^*$ under Ramsey plan (41) is constant:

$$C_{0,t}^* = \left(\sqrt{1 + 4b_0\Phi_0^*(1 + \Phi_0^*)} - 1 \right) / (2(1 + \Phi_0^*)).$$

The equilibrium interest rate $r_{0,t}^*$ equals ρ for all $t \geq 0$.

Peaks, troughs, and inflection points of the primary surplus process $S_{0,t}^* - b_{0,t}$ coincide with those of $G_{0,t}$ because

$$\frac{dG_t}{dt} = -\frac{\pi}{2}\theta_G \sin\left(\frac{\pi}{2}t\right); \quad \frac{d^2G_t}{dt^2} = -\left(\frac{\pi}{2}\right)^2 \theta_G \cos\left(\frac{\pi}{2}t\right),$$

and

$$\frac{d(S_{0,t}^* - b_{0,t})}{dt} = -C_{0,t} \frac{dG_t}{dt}; \quad \frac{d^2(S_{0,t}^* - b_{0,t})}{dt^2} = -C_{0,t} \frac{d^2G_t}{dt^2}.$$

Since under a constant interest rate $r_{0,t}^* = \rho$ cumulative liabilities Π_t^* equal the present

value of the net-of-interest surplus $S(C_{0,t}^*) - b_{0,t}$, it is also a periodic function of time t . But Π_t^* peaks and troughs at $S_{0,t}^* = b_{0,t}$, since

$$\frac{d\Pi_t^*}{dt} = 0 \Leftrightarrow (b_{0,t} - S_{0,t}^*) = 0. \quad (\text{A-18})$$

Because $U_{C,t}$ is constant over time, the state variable $X_t^* = \Pi_t^* U_{C,t}^*$ is also a periodic function. Its peaks and troughs coincide with those of Π_t^* .

Table 3: Peaks and troughs in Figure 8.

	$t = 0$	$t = 2$
Government spending G_t	Peak	Trough
Tax rate $\tau_{0,t}^*$	constant	
Interest rate $r_{0,t}^*$	constant	
$S(C_{0,t}^*) - b_{0,t}$	Trough	Peak